main thing going on in chapters 15-16:

- use line bundles to understand proj. schemes better] start

Chapter 15: contains various general results about qcoh sheaves on Proj $S_\bullet$, which we mostly won't worry about proving.] today.

15.3 is a bit of an aside, we will delay it a bit.

Old story about Proj $S_\bullet$:

graded ring $S_\bullet$ f.g. index 1 $\rightsquigarrow$ scheme $X = \operatorname{Proj} S_\bullet$

closed embedding $\iota: X \hookrightarrow \mathbb{P}^m_{S_0}$

New story: also get a line bundle $\mathcal{O}_{\operatorname{Proj} S_\bullet}(1) = \mathcal{O}(1)$ on X,

$\overset{\shortparallel}{\iota^* \mathcal{O}_{\mathbb{P}^m_{S_0}}(1)}$

Should think of this as a more intrinsic way of giving the data of the closed embedding (most of)

Today: classify qcoh sheaves on Proj S.

Idea: qcoh. sheaves on Spec $A \longleftrightarrow$ A-modules

qcoh sheaves on Proj $S_\bullet \overset{?}{\longleftrightarrow} S_\bullet$-modules.

Def: A graded $S_\bullet$-module is an $S_\bullet$-module $M_\bullet$ with its own grading $M_\bullet = \bigoplus_{i \in \mathbb{Z}} M_i$ s.t. the two gradings are capatible, i.e.

$f \in S_i$ and $x \in M_j$ then $fx \in M_{i+j}$.

Example: Any homog. ideal in $S_\bullet$ is a graded $S_\bullet$-module.

Def: Given a graded $S_\bullet$-module $M_\bullet$, there is a qcoh sheaf $\widetilde{M_\bullet}$ on Proj $S_\bullet$, defined by $\widetilde{M_\bullet}(D(f)) = (M[\tfrac{1}{f}])_0$ (which is a module over $(S[\tfrac{1}{f}])_0$, and $\mathcal{O}_{\mathrm{Proj}\, S_\bullet}(D(f)) = (S[\tfrac{1}{f}])_0$)

$$\boxed{\text{alt. } \widetilde{M_\bullet}|_{D(f)} = \widetilde{(M[\tfrac{1}{f}])_0}}$$

Example: If $M_\bullet = I_\bullet$ is a homog. ideal, then $\widetilde{I_\bullet}$ is the ideal sheaf of the closed subscheme

$$\mathrm{Proj}\,(S/I)_\bullet \hookrightarrow \mathrm{Proj}\, S_\bullet \,.$$

Can check: if $\varphi: M_\bullet \to N_\bullet$ is a homomorphism, then get $\varphi_*: \widetilde{M_\bullet} \to \widetilde{N_\bullet}$. In other words, this construction defines a functor

$$\{\text{graded } S_\bullet\text{-modules}\} \longrightarrow \left\{\begin{matrix}\text{qcoh sheaves}\\ \text{on Proj } S_\bullet\end{matrix}\right\}$$

Q: Is this an equiv. of categories?

A: No, but somewhat close:

Just as $\operatorname{Proj} S \cong \operatorname{Proj} S' \not\Rightarrow S \cong S'$,

$$\widetilde{M_\bullet} \cong \widetilde{M'_\bullet} \not\Rightarrow M_\bullet \cong M'_\bullet$$

<u>Example</u>: if $M_\bullet \to M'_\bullet$ is an isomorphism in all suff. large degrees, then

$$\widetilde{M_\bullet} \cong \widetilde{M'_\bullet}.$$

these issues can be addressed by requiring that M, M' be <u>saturated</u> (to be defined later, not important).

As a preview of "saturated", it will require that

$$\boxed{M_0 \xrightarrow{\sim} \widetilde{M_\bullet}(\operatorname{Proj} S_\bullet)}$$

<u>Def</u>: (shifts of graded modules): Let $M_\bullet$ be a graded $S_\bullet$-module and let $n \in \mathbb{Z}$. Then define a new graded module $M(n)_\bullet$ by

$$M(n)_m := M_{n+m}.$$

<u>Prop/def</u>: Suppose $S_\bullet$ is gen. in degree 1. Then the qcoh sheaf

$$\mathcal{O}_{\operatorname{Proj} S_\bullet}(n) := \widetilde{S(n)_\bullet} \text{ is a line bundle.}$$

(immediately get $\mathcal{O}_{\operatorname{Proj} S_\bullet}(0) = \mathcal{O}_{\operatorname{Proj} S_\bullet}$)

<u>Pf</u>: Let $f \in S_1$. Then

$$\widetilde{S(n)_\bullet}(D(f)) = \left(S(n)\left[\tfrac{1}{f}\right]\right)_0 = \left(S\left[\tfrac{1}{f}\right]\right)_n$$

is a free module of rank 1 over $\left(S\left[\frac{1}{f}\right]\right)_0$, with gen. f^n. This is saying that $\widetilde{S(n)_\bullet}\big|_{D(f)} \cong \mathcal{O}_{D(f)}$, and such $D(f)$ form an open cover. $\blacksquare$

Can check: $\mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(m+n)$,

so we have a group homomorphism

$$\left.\begin{array}{l} \mathbb{Z} \longrightarrow \operatorname{Pic}(\operatorname{Proj} S_\bullet) \\ n \longmapsto \mathcal{O}(n) \end{array}\right] \text{isom for } S_\bullet = k[x_0, \ldots, x_m]$$

Thinking about $\mathcal{O}(1)$ geometrically:

If $\operatorname{Proj} S_\bullet$ is normal, then the line bundle $\mathcal{O}_{\operatorname{Proj} S_\bullet}(1)$ must correspond to a Weil divisor (up to equiv.).

So we can take a rational section s of $\mathcal{O}_{\operatorname{Proj} S_\bullet}(1)$.

Any element $f \in S_1$ will give a global section of $\mathcal{O}(1)$.

(in general, have a map

$$S_1 \longrightarrow \Gamma(\operatorname{Proj} S_\bullet, \mathcal{O}_{\operatorname{Proj} S_\bullet}(1)))$$

So we want the divisor (of zeroes) $\operatorname{div}(f) \geq 0$. Or equiv, could take the Weil divisor corresp. to the effective Cartier divisor $V(f)$.

If $S_\bullet$ is f.g. (in deg 1), then we can write

$S_\bullet = S_0[x_0, \dots, x_m]/I$ for some homog. ideal I,

corresponding to an embedding of $\mathrm{Proj}\, S_\bullet$ in some $\mathbb{P}^m$.

Then the divisor in question ($\mathrm{div}(f)$) is then the intersection of $\mathrm{Proj}\, S$ with some hyperplane in $\mathbb{P}^m$.

($X = \mathrm{Proj}\, S_\bullet$)

<u>Slogan</u>: "$\mathcal{O}_X(1) \cong \mathcal{O}_X(H \cap X)$" ← hyperplane.

more generally: $\mathcal{O}_X(d) \cong \mathcal{O}_X(Y_d \cap X)$ ← deg d hypersurface.

Generally expect $\mathcal{O}_X(n)$ doesn't have global sections for $n<0$.

But can still write $\mathcal{O}_X(-1) \cong \mathcal{O}_X(-[H \cap X])$.

Important def: Suppose $S_\bullet$ is gen in deg 1 and $\mathcal{F}$ is a qcoh sheaf on Proj $S_\bullet$. Then the (Serre) twist of $\mathcal{F}$ by $n \in \mathbb{Z}$ is

$$\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_{\mathrm{Proj}\, S_\bullet}(n).$$

(infinite family of modifications of $\mathcal{F}$)

~~Can~~ check: $\widetilde{M_\bullet(n)} \cong \widetilde{M}(n)$.

Finishing the story of saturated modules:

$S_\cdot$ = f.g. graded ring gen. in deg 1.

$X = \mathrm{Proj}\, S_\cdot$

$M_\cdot$ = graded $S_\cdot$-module.

<u>Def</u>: The <u>saturation of $M_\cdot$</u> is a new graded $S_\cdot$-module $\Gamma M_\cdot$ defined by

$$\Gamma M_n := \Gamma(X, \widetilde{M_\cdot}(n)),$$

with module structure is given via the maps

$$S_m \longrightarrow \Gamma(X, \mathcal{O}(m)) \text{ and } \otimes.$$

There is a natural module homomorphism

$$M_\cdot \longrightarrow \Gamma M_\cdot$$

<u>Def</u>: A module $M_\cdot$ is <u>saturated</u> if it is equal to its saturation, i.e. $M_\cdot \xrightarrow{\sim} \Gamma M_\cdot$.

<u>Thm</u>: There is an equivalence of categories

$$\{\text{saturated } S_\cdot\text{-modules}\} \xrightarrow{\sim} \left\{\begin{matrix}\text{qcoh sheaves} \\ \text{on } X\end{matrix}\right\}$$

$$\bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n)) \longleftarrow\!\shortmid\; \mathcal{F}$$

Q: What happens when you apply saturation to $S_\bullet$ as a module over itself.

$$\boxed{\Gamma S_\bullet := \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}(n))}$$

This is a graded ring. (since $\mathcal{O}(m)\otimes\mathcal{O}(n)\cong\mathcal{O}(m+n)$).

Hope: this might be some canonical form for the graded ring $S_\bullet$, i.e. address our issues with $\mathrm{Proj}\, S_\bullet \cong \mathrm{Proj}\, S'_\bullet \not\Rightarrow S_\bullet \cong S'_\bullet$

Almost true: have $\mathrm{Proj}\, \Gamma S_\bullet \cong \mathrm{Proj}\, S_\bullet$, and can think of this as a way of recovering $S_\bullet$ from the data

$$(X, \mathcal{O}(1)).$$

(Warning: $\mathrm{Proj}\, S_\bullet \cong \mathrm{Proj}\, S'_\bullet \not\Rightarrow \mathcal{O}_{\mathrm{Proj}\, S_\bullet}(1) \cong \mathcal{O}_{\mathrm{Proj}\, S'_\bullet}(1)$).