(apologies in advance – family emergency, potentially have to leave abruptly)

<u>No office hours today.</u>

Last time: (assuming <u>X Noetherian and normal</u>) (and X irred. for convenience)

$$\begin{tikzcd}
\{(\mathcal{L}, s)\}/_{\text{iso}} \arrow[r, hook, bend left, "\mathrm{div}"] & \mathrm{Weil}\, X \\
 & \mathrm{im}(\mathrm{div}) \arrow[u, phantom, sloped, "\subseteq"] \arrow[ul, bend left, "\sim"', "{(\mathcal{O}_X(-),\, 1)}"]
\end{tikzcd}$$

<u>Defs:</u> 1) $D \in \mathrm{Weil}\, X$ is <u>principal</u> if $D = \mathrm{div}(f)$ for $f \in K(X)^\times$ (i.e. a rational section of $\mathcal{O}_X$)

2) $D \in \mathrm{Weil}\, X$ is <u>locally principal</u> if there is an open cover of X by opens U with $D|_U$ principal.

<u>Lemma:</u> $\mathrm{im}(\mathrm{div}) = \{\text{locally principal } D \in \mathrm{Weil}\, X\}$.

$\subseteq$: follows from $\mathcal{L}$ being locally iso to $\mathcal{O}_U$.

<u>want</u>: if $D \in \mathrm{Weil}\, X$ is loc. principal, then $D = \mathrm{div}(s)$ for some rat'l. section s of some line bundle.

1) Check that $\mathcal{O}_X(D)$ is a line bundle
   - follows from $\mathcal{O}_U(\mathrm{div}(f)) \cong \mathcal{O}_U$ for $\underset{\text{open}}{U \subseteq X}$, $f \in K(U)$.

2) Check that the rat'l. section 1 of $\mathcal{O}_X(D)$ has divisor D. (Again, check locally).

At this point we can say

$$\mathrm{Pic}(X) = \{\mathcal{L}\}/\mathrm{iso} \cong \frac{\{(\mathcal{L}, s)\}/\mathrm{iso}}{\{(\mathcal{O}_X, s)\}/\mathrm{iso}}$$

$$\underset{\mathrm{div}}{\cong} \frac{\{\text{locally principal Weil divisors}\}}{\{\text{principal Weil divisors}\}}.$$

<u>Conditions</u> under which every divisor is (locally) principal:

<u>Lemma</u>: Let A be a UFD (unique factorization domain: any elt can be written uniquely as a product of irred. elements, up to units). Then every codim 1 prime ideal in A is principal.

(<u>Pf</u>: given a nonzero prime ideal $\mathfrak{p} \subset A$, find an irred element $x \in \mathfrak{p}$ (by factoring any elt in $\mathfrak{p}$), and then $0 \subset (x) \subseteq \mathfrak{p} \underset{\mathfrak{p} \text{ has height } 1}{\Longrightarrow} \mathfrak{p} = (x)$. $\blacksquare$)

<u>Cor 1</u>: If A is a UFD (e.g. $k[t_1, \ldots, t_n]$), then every Weil divisor in $\mathrm{Spec}\, A$ is principal, and thus $\mathrm{Pic}(\mathrm{Spec}\, A) = 0$.

Cor 2: If X is factorial (i.e. all stalks $\mathcal{O}_{X,p}$ are UFDs), then every $D \in \mathrm{Weil}\, X$ is locally principal.

Note: regular $\Rightarrow$ factorial $\Rightarrow$ normal $\Rightarrow$ regular in codim 1.

Def: The (Weil) class group of X is

$$Cl\, X = \mathrm{Weil}\, X \Big/ \left\{ \begin{array}{c}\text{principal Weil divisors} \\ \text{in } X \end{array} \right\}.$$

($D_1 = D_2$ in $Cl\, X$: "linearly equivalent")

Summary: If X is Noetherian and factorial (e.g. regular),

$$\begin{tikzcd}
\{(\mathcal{L}, s)\}/\mathrm{iso} \arrow[r, leftrightarrow, "\sim"] \arrow[d] & \mathrm{Weil}\, X \arrow[d] \\
\mathrm{Pic}\, X = \{\mathcal{L}\}/\mathrm{iso} \arrow[r, leftrightarrow, "\sim"] & Cl\, X
\end{tikzcd}$$

$$\mathcal{L} \longmapsto \mathrm{div}(s) \text{ for any nonzero rat. section } s \text{ on } \mathcal{L}$$

$$\mathcal{O}_X(D) \longleftarrow\!\shortmid\; D$$

Computations of Pic X / Cl X:

1) $\mathrm{Pic}(\mathbb{A}^n_k) = 0$ by the UFD criterion (since $k[t_1, \ldots, t_n]$ is a UFD)

2) two approaches to $\mathrm{Pic}(\mathbb{P}^n_k)$:

a) Let $H \in \mathrm{Cl}(\mathbb{P}^n_k)$ be the class of the hyperplane $V(x_0)$. Then any irred divisor of $\mathbb{P}^n_k$ is of the form $V(f)$ for homog. f of deg $d > 0$, and we have

$$\mathrm{div}\left(\frac{f}{x_0^d}\right) = [V(f)] - d[V(x_0)]$$

is principal, so $[V(f)] = dH$ in $\mathrm{Cl}(\mathbb{P}^n_k)$.

Thus H generates $\mathrm{Cl}(\mathbb{P}^n_k)$.

But we've used all the relations coming from principal divisors, so $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}H$.

b) More general tool: Suppose Y is an irred. divisor in X. Then there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \operatorname{Weil} X \longrightarrow \operatorname{Weil}(X \setminus Y) \longrightarrow 0$$
$$1 \longmapsto [Y] \qquad\qquad |_{X \setminus Y}$$
,

so

$$\mathbb{Z} \longrightarrow \operatorname{Cl} X \longrightarrow \operatorname{Cl}(X \setminus Y) \longrightarrow 0$$
$$1 \longmapsto [Y]$$

is also exact ("excision")

Apply to $X = \mathbb{P}^n_k$, $Y = \{x_0 = 0\}$, so $X \setminus Y \cong \mathbb{A}^n_k$, and immediately get $\operatorname{Cl} X$ is generated by $H = [Y]$ as before.

An independent argument that $dH \neq 0$ in $\operatorname{Cl} X$ for all $d > 0$: check that $\mathcal{O}_{\mathbb{P}^n_k}(dH) \left(\cong \mathcal{O}_{\mathbb{P}^n_k}(d) \right)$ is nontrivial for $d \neq 0$, e.g. by computing global sections:

$$\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(dH)) \cong k^{a_d}, \text{ where } a_d = 1 \Leftrightarrow d = 0.$$

3) More excision:

Let Z be an irred hypersurface of degree d in $\mathbb{P}^n_k$,

i.e. $Z = V(f)$ for f homog of deg d.

Then excision gives

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & Cl(\mathbb{P}^n_k) & \longrightarrow & Cl(\mathbb{P}^n_k \setminus Z) & \to & 0, \\ & & \wr\!\!\mid\!\mid & & & & \\ 1 & \longmapsto & d \quad \mathbb{Z} & & & & \end{array}$$

so $\mathrm{Pic}(\mathbb{P}^n_k \setminus Z) \cong \mathbb{Z}/d\mathbb{Z}$, so it has torsion for $d > 1$.

Note: $\mathrm{Pic}(Z)$ is much more complicated, excision doesn't help us here directly.

Recall: given a morphism $\pi: X \to Y$, we can pull back qcoh sheaves from Y to X. The pullback of a line bundle is a line bundle, so get a group homomorphism $\pi^*: \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$.

Problem set will have a couple more examples of computing small Picard group. These examples might lead you to believe that $\mathrm{Pic}(X)$ tends to be "small", e.g. a f.g. abelian group.

This is <u>false</u>: in general $\mathrm{Pic}(X)$ is not f.g.

(e.g. X a regular plane curve of degree ≥ 3)

(Recall: $\mathrm{Pic}(X) \cong \underbrace{\mathrm{Weil}(X)}_{\text{not f.g.}} / \{\text{principal divs}\}$.