<u>Recall</u>: If $\mathcal{F}$ is a qcoh sheaf on X and $p \in X$, then the fiber of $\mathcal{F}$ at p is

$$\mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} k_p .$$

<u>Def</u>: If $s \in \mathcal{F}(U)$ is a section, its <u>value at p</u> is its image in $\mathcal{F}|_p$ (denoted $s(p)$ or $s|_p$).

<u>Def</u>: The <u>rank of $\mathcal{F}$ at p</u> is $\dim_{k_p} \mathcal{F}|_p$.

<u>Examples</u>:

1) A line bundle has rank 1 at every point. More generally, a vector bundle of rank r has rank r at every point.

2) $X = \operatorname{Spec} k[t]$, $\mathcal{F} = \widetilde{k[t]/(t)} =$ skyscraper sheaf with value k at the origin

$$\mathcal{F} = \iota_* \mathcal{O}_{\operatorname{Spec} k} \text{ for } \iota : \operatorname{Spec} k \to \operatorname{Spec} k[t]$$
$$0 \leftarrow\!\shortmid\, t,$$

has rank 1 at the origin, 0 everywhere else.

3) Let $\mathcal{I}$ be the ideal sheaf corresponding to $\{\text{origin}\} \subset \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$
as a reduced closed subscheme.

So $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{A}^2_k}$ consisting of functions vanishing at the origin.

So $\mathcal{I} = \widetilde{(x,y)}$ (here (x,y) is a $k[x,y]$-module)

Can check: $\mathcal{I}$ has rank 2 at the origin, but rank 1 everywhere else.

4) Rank is additive/multiplicative for $\oplus$/$\otimes$, but not well-behaved for exact sequences in general:

e.g. on $\mathbb{A}^2_k$: $0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{A}^2_k} \to i_* \mathcal{O}_{\operatorname{Spec} k} \to 0$

* rank at origin:	0	2	1	1	0
rank elsewhere: (closed points)	0	1	1	0	0

("$M \rightsquigarrow M/\mathfrak{m}_p M$ is not an exact functor")

fibers of sheaves in an exact sequence are not necessarily exact.

Observation: in the examples so far, rank is an upper semicontinuous function on the points of X, i.e.

$\{p \in X \mid \mathrm{rank}_{\mathcal{F}}(p) \geq n\}$ is a closed subset of X.

(similar to results about dimensions of fibers of morphisms)

This observation is clearly not true for general qcoh $\mathcal{F}$, e.g. we could take

$$\mathcal{F} = \bigoplus_{\substack{p \in \mathbb{A}^1_{\mathbb{C}} \\ \text{closed}}} \iota_{p*} \mathcal{O}_{\mathrm{Spec}\, k_p} \quad (\text{for } \iota_p : \mathrm{Spec}\, k_p \to \mathbb{A}^1_{\mathbb{C}})$$

Suggests we want some finite generation assumption on $\mathcal{F}$.

We'd also like to say that

$\mathcal{F}$ is a rank r vector bundle

$\iff$ $\mathcal{F}$ has rank r at every point, but again easy to come up with a counterexample: $\mathcal{F} = \widetilde{k(t)/k[t]}$ on $\mathbb{A}^1_k$

Def: A qcoh sheaf $\mathcal{F}$ on X is <u>finite type</u> if for every affine open $\operatorname{Spec} A \subset X$, $\mathcal{F}(\operatorname{Spec} A)$ is a f.g. A-module.

<u>Issue</u>: (for people working with schemes that are not locally Noetherian):

f.g. A-modules do not form an abelian category if A is not Noetherian, so neither do finite type qcoh sheaves (kernels might not be finite type).

<u>Defs</u>: An A-module M is <u>coherent</u> if it is f.g. and any morphism $A^n \to M$ has f.g. kernel.

A qcoh sheaf $\mathcal{F}$ on X is <u>coherent</u> if for every affine open $\operatorname{Spec} A \subset X$, $\mathcal{F}(\operatorname{Spec} A)$ is a coherent A-module.

(Remark: on a loc. Noetherian scheme X, coherent sheaf $\Longleftrightarrow$ finite type qcoh sheaf.)

(I will follow Vakil and use "finite type" when that is sufficient for the purposes at hand.)

(coherent $\Longrightarrow$ finite type in general)

<u>Can prove</u>: "finite type (or coherent) is an affine-local property",

i.e. if $\mathcal{F}(\operatorname{Spec} A)$ is a f.g. A-module for affine opens covering X, then $\mathcal{F}$ is finite type.

<u>Cor</u>: Vector bundles (of rank $r < \infty$) are coherent.

(since $\mathcal{O}_X^{\oplus r}$ is coherent).

Geometric Nakayma's Lemma:

X = scheme

$\mathcal{F}$ = finite type qcoh sheaf on X

$p \in U \underset{\text{open}}{\subseteq} X$

$s_1, \dots, s_n \in \mathcal{F}(U)$ such that

$s_1(p), \dots, s_n(p)$ generate $\mathcal{F}|_p$ (as a k_p-vector space).

Then there is an affine open nhood

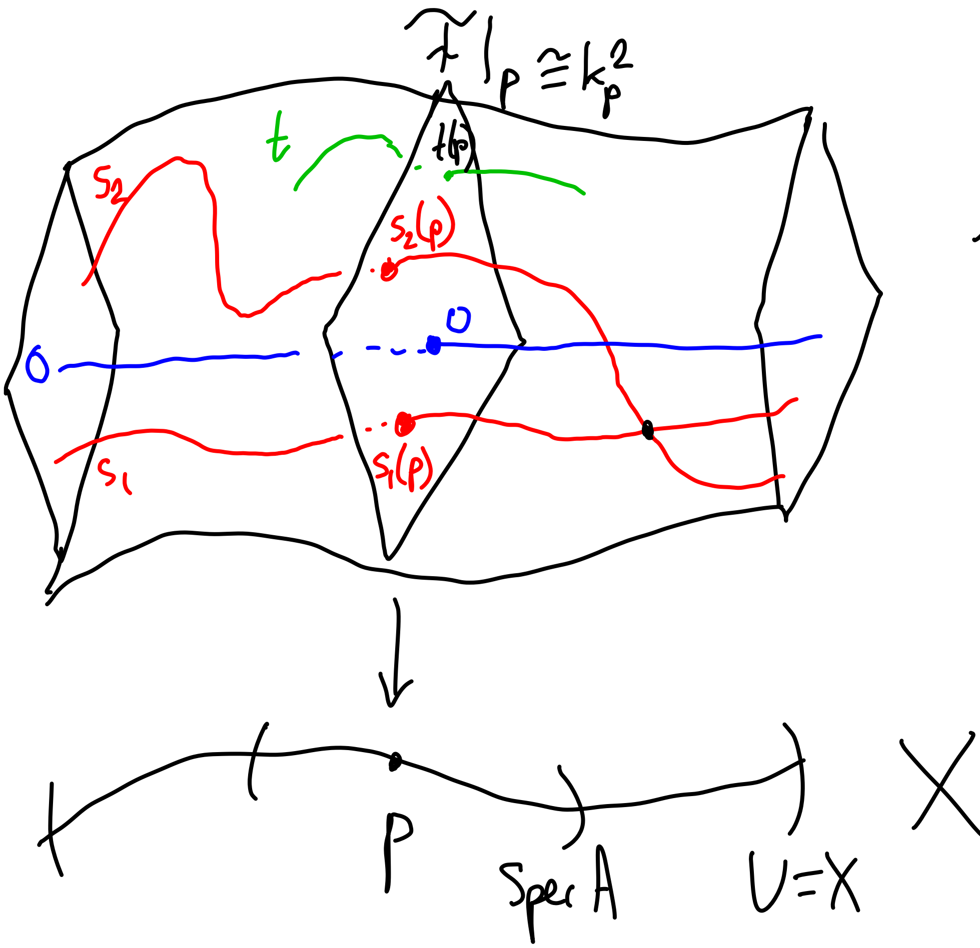
$p \in \operatorname{Spec} A \subseteq U$ s.t.

$s_1|_{\operatorname{Spec} A}, \dots, s_n|_{\operatorname{Spec} A}$ generate $\mathcal{F}(\operatorname{Spec} A)$ as an A-module.

(Hence $s_1, \dots, s_n$ gen all the stalks and fibers at points $\in \operatorname{Spec} A$.)

Cor 1: $\operatorname{rank}_{\mathcal{F}}(p)$ is upper semicontinuous.

Pf: If n generators suffice for $\mathcal{F}|_p$, then they also suffice for $\mathcal{F}|_q$ for $q \in \operatorname{Spec} A \ni p$, so for $q \in \operatorname{Spec} A$, $\operatorname{rank}_{\mathcal{F}}(q) \leq \operatorname{rank}_{\mathcal{F}}(p)$.



$\mathcal{F}$ = finite type qcoh sheaf on X

$t \in \mathcal{F}(\operatorname{Spec} A)$ is a linear combination of $s_1|_{\operatorname{Spec} A}, s_2|_{\operatorname{Spec} A}$ with coefficients in $\mathcal{O}_X(\operatorname{Spec} A) = A$

<u>Nakayama's Lemma:</u> Suppose (A, m) is a local ring and M is a f.g. A-module. Suppose $m_1, \ldots, m_n \in M$ generate M/mM (as a $k = A/m$ -vector space).

Then $m_1, \ldots, m_n$ generate M.

This corresponds precisely to (in Geom. Nakayama's Lemma)

if $s_1, \ldots, s_n$ generate $\mathcal{F}|_p$, then

$s_1, \ldots, s_n$ generate $\mathcal{F}_p$.

The remaining step, that

if $s_1, \ldots, s_n$ generate $\mathcal{F}_p$, then they generate $\mathcal{F}(\operatorname{Spec} A)$ for some affine open $\operatorname{Spec} A \ni p$,

follows from observing that only finitely many elements are inverted in the statement "$s_1, \ldots, s_n$ generate $\mathcal{F}_p$".

so instead of using M_p, can use $M[\frac{1}{f}]$ for some f.

(so pass from $\operatorname{Spec} A'$ to $\operatorname{Spec} A'[\frac{1}{f}]$)

<u>Cor 2</u>: If X is <u>reduced</u> and $\mathrm{rank}_{\mathcal{F}}(p)$ is constant (and $\mathcal{F}$ is a finite type qcoh sheaf on X), then $\mathcal{F}$ is a vector bundle of rank $\mathrm{rank}_{\mathcal{F}}(p)$.

<u>Pf.</u> Suppose $p \in X$; we want to find affine open $\operatorname{Spec} A \ni p$ with $\mathcal{F}(\operatorname{Spec} A) \cong A^r$.

Choose $s_1, \ldots, s_r$ defined near p with values in $\mathcal{F}|_p \cong k_p^r$ giving a basis.

So we can find $\operatorname{Spec} A \ni p$ where $s_1, \ldots, s_r$ are defined and generate $\mathcal{F}(\operatorname{Spec} A)$

If $s_1, \ldots, s_r$ have no relations in this module, then we would be done. So assume
$f_1 s_1 + \cdots + f_r s_r = 0$ for $f_1, \ldots, f_r \in A = \mathcal{O}_X(\operatorname{Spec} A)$; not all 0.

So some $f_i \neq 0$, and since X is reduced, this means $f_i(q) \neq 0$ for some $q \in \operatorname{Spec} A$.

So in $\mathcal{F}|_q$, we have a nontrivial relation

$$f_1(q)s_1(q) + \cdots + f_r(q)s_r(q) = 0,$$

but $s_1(q), \ldots, s_r(q)$ generate $\mathcal{F}|_q$, so

$\operatorname{rank}_{\mathcal{F}}(q) < r$. $\Rightarrow\Leftarrow$