<u>Last time:</u> line bundles $\subseteq$ vector bundles $\subseteq$ qcoh sheaves
loc. $\mathcal{O}_U$, loc. $\mathcal{O}_U^{\oplus I}$, $\tilde{M}$ on Spec A,
M is an A-module.

Category of qcoh sheaves on Spec A is equivalent to the category of A-modules.
(on non-affine X, "gluing" modules over diff. rings)

Line bundles on Spec A are more interesting
– correspond to certain "locally trivial" A-modules
(invertible fractional ideals)

<u>Examples!</u>

1) What are the line bundles on $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$?

$M = k[t]$-module
$f_1, \dots, f_m \in k[t]$ with $\boxed{(f_1, \dots, f_m) \ni 1}$
($\Longleftrightarrow \{D(f_i)\}$ cover $\mathbb{A}^1_k$)
with $M[\frac{1}{f_i}] \cong k[t][\frac{1}{f_i}]$ (as a $k[t][\frac{1}{f_i}]$-mod)
f.g.

$\rightsquigarrow$ M is f.g. $k[t]$-module

$\Longrightarrow M \cong \bigoplus_{j=1}^{\ell} k[t]/(g_j)$

But we still need $M[\frac{1}{f_i}] \cong k[t][\frac{1}{f_i}]$

If some $g_j \neq 0$, then choose f_i not a multiple of g_j to get a contradiction.

Conclude $M \cong k[t]$

<u>Prop</u>: All line bundles on $\mathbb{A}^1_k$ are trivial ($\cong \mathcal{O}_{\mathbb{A}^1_k}$).

(But $\operatorname{Spec} \mathbb{Z}[\sqrt{-5}]$ will have nontrivial line bundles)

(Can generalize the above argument to show that all line bundles on $\operatorname{Spec} A$ are trivial if A is a PID (or indeed if A is a UFD).)

Example 2: What about line bundles on $\mathbb{P}^1_k$?

$\mathcal{L}$ line bundle on $\mathbb{P}^1_k = \operatorname{Proj} k[x,y]$

$$\operatorname{Proj} k[x,y] = D(x) \cup D(y), \quad D(x) \cong \mathbb{A}^1, \quad D(y) \cong \mathbb{A}^1.$$

$\mathcal{L}|_{D(x)}$ is a line bundle, so isomorphic to

$$\mathcal{O}_{D(x)} = \widetilde{k\left[\tfrac{y}{x}\right]}$$

Similarly $\mathcal{L}|_{D(y)} \cong \widetilde{k\left[\tfrac{x}{y}\right]}$.

Gluing these trivial line bundles together means choosing an isom of

$$\mathcal{O}_{D(x)}\big|_{D(xy)} \text{ with } \mathcal{O}_{D(y)}\big|_{D(xy)}$$

Convert into modules: want an automorphism of $k\left[\tfrac{x}{y}, \tfrac{y}{x}\right]$ as a module over itself.

These automorphisms are just invertible elements of $k[\frac{x}{y}, \frac{y}{x}]$, i.e. elements $c \cdot \left(\frac{x}{y}\right)^n$ for $c \in k^*$, $n \in \mathbb{Z}$.

$$
\begin{tikzcd}
k\left[\frac{x}{y}\right] \arrow[loop right, "k^*"] \arrow[d] & k\left[\frac{y}{x}\right] \arrow[loop right, "k^*"] \arrow[d] \\
k\left[\frac{x}{y}, \frac{y}{x}\right] \arrow[r, "\sim", "\boxed{* \, c \cdot \left(\frac{x}{y}\right)^n}"'] & k\left[\frac{y}{x}, \frac{x}{y}\right]
\end{tikzcd}
$$

"transition function" can eliminate by using freedom of choice of isom.

<u>Conclusion</u>: The isomorphism classes of line bundles on $\mathbb{P}^1_k$ are in bijection with $\mathbb{Z}$.

Transition functions:

$\mathcal{L}$ = line bundle on X

chosen trivializations $\varphi_i \; \mathcal{L}|_{U_i} \longrightarrow \mathcal{O}_{U_i}$

on some open cover $\{U_i\}$.

Then $\mathcal{O}_{U_i \cap U_j} \xrightarrow{\varphi_i^{-1}|_{U_i \cap U_j}} \mathcal{L}|_{U_i \cap U_j} \xrightarrow{\varphi_j|_{U_i \cap U_j}} \mathcal{O}_{U_i \cap U_j}$

$\rho_{ij} = \varphi_j|_{U_i \cap U_j} \circ \varphi_i^{-1}|_{U_i \cap U_j}$ is an automorphism

of $\mathcal{O}_{U_i \cap U_j}$, so $\rho_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*$.

The data of $(\rho_{ij})_{i,j}$ (the <u>transition functions</u> of the trivializations chosen)

precisely lets us recover the line bundle $\mathcal{L}$.

$$\left\{ \begin{array}{c} \mathcal{L} \text{ line bundle on } X \\ \text{choice of } \underline{\text{open cover } \{U_i\}} \\ \text{and } \underline{\text{trivializations } \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}} \end{array} \right\} \longleftrightarrow \left\{ (U_i), (\rho_{ij}) \right\}.$$

Operations on qcoh sheaves (basically anything that makes sense with modules)

- abelian category operations: $\oplus$, ker, coker
  - on affine opens, correspond to same operation on modules (operations commute with the functor $\mathcal{F} \longrightarrow \mathcal{F}(\operatorname{Spec} A)$)

- other module operations: $\otimes$, Hom
  - commute with localization ($\operatorname{Hom}_A(M,N)[\frac{1}{f}] \cong \operatorname{Hom}_{A[\frac{1}{f}]}(M[\frac{1}{f}], N[\frac{1}{f}])$)
  - $(\mathcal{F} \otimes \mathcal{G})(\operatorname{Spec} A) = \mathcal{F}(\operatorname{Spec} A) \otimes_A \mathcal{G}(\operatorname{Spec} A)$.
  - Hom corresponds to the general ("Sheaf Hom") notion $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \operatorname{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$.

Remark: If $\mathcal{F}, \mathcal{G}$ are vector bundles of ranks m, n, then $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ will also be vector bundles of ranks $m+n$, mn, mn (corresp. to same statement about $M^{\oplus n}$.

In particular:

- If $\mathcal{L}_1, \mathcal{L}_2$ are line bundles, then so is
$$\left.\mathcal{L}_1 \otimes \mathcal{L}_2 .\right]$$ multiply transition functions $p_{ij} p'_{ij}$
- If $\mathcal{L}$ is a line bundle, then so is
$$\left.\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X).\right]$$ invert transition functions $1/p_{ij}$
- If $\mathcal{L}$ is a line bundle, then
$$\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X .$$

<u>Very important def</u>: Let X be a scheme. Then
$$\mathrm{Pic}(X) := \{\text{line bundles on } X\}/\text{isom}$$
is an abelian group under $\otimes$ with identity $\mathcal{O}_X$ and inverses given by $\mathcal{L} \mapsto \mathcal{L}^\vee$.

This is called the <u>Picard group</u> of X.

We've seen: $\mathrm{Pic}(\mathbb{A}^1_k) = 0$
$\mathrm{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$.

Observation: Given two sections

$$s_1 \in \mathcal{L}_1(X), \quad s_2 \in \mathcal{L}_2(X),$$

there is a natural section

$$s_1 s_2 = s_1 \otimes s_2 \in (\mathcal{L}_1 \otimes \mathcal{L}_2)(X).$$

(construct by gluing on affine opens, since

$$(\mathcal{L}_1 \otimes \mathcal{L}_2)(\operatorname{Spec} A) = \mathcal{L}_1(\operatorname{Spec} A) \otimes \mathcal{L}_2(\operatorname{Spec} A).$$

A couple more operations:

- pushforward: Recall if $\pi: X \to Y$ and $\mathcal{F}$ is a sheaf on X, then

$$(\pi_* \mathcal{F})(U) = \mathcal{F}(\pi^{-1}(U)).$$

<u>Prop</u>: If $\mathcal{F}$ is a qcoh sheaf on X and $\pi: X \to Y$ is qcqs, then $\pi_* \mathcal{F}$ is a qcoh sheaf on Y.

<u>Pf</u>: (essentially generalization of "Qcqs Lemma", see 13.3.E in Vakil).

Nice special case: if $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, B$, so
A is a B-algebra, then pushforward corresponds to the forgetful functor

$$\mathrm{Mod}_A \to \mathrm{Mod}_B.$$

<u>Pullback</u>: Given a morphism $\pi: X \to Y$ and a qcoh sheaf $\mathcal{F}$ on Y, there is a qcoh sheaf $\pi^* \mathcal{F}$ on X.

(alg) Idea of construction: If $\pi: \mathrm{Spec}\, A \to \mathrm{Spec}\, B$,

$$\text{then } \pi^* \widetilde{M} = \widetilde{\underbrace{M \otimes_B A}}.$$

(geom) Idea of construction: If $\mathcal{F}$ is the sheaf of sections of some morphism $Z \to Y$,
then $\pi^* \mathcal{F}$ should be the sheaf of sections of $Z \times_Y X \to X$.

Nice feature: If $\mathcal{F}$ is a rank n vector bundle, then so is $\pi^* \mathcal{F}$.

(So $\pi: X \to Y$ will induce a map $\pi^*: \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ by pullbacks).

Next time: local properties (at points) of qcoh sheaves,

e.g.: If $\mathcal{F}$ is a qcoh sheaf on X, then there are stalks $\mathcal{F}_p$, which will be modules over $\mathcal{O}_{X,p}$.

Def: The fiber of $\mathcal{F}$ at p is

$$\mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} k_p = \mathcal{F}_p / m_p \mathcal{F}_p ,$$

which is a k_p-vector space.

Intuition (but be careful): $\mathcal{F}$ is a "vector bundle with fibers of varying dimension", and

$\dim_{k_p} \mathcal{F}|_p$ is this dimension at a point.

Example: $\widetilde{k[t]/t}$ as a sheaf on $\operatorname{Spec} k[t]$ has fiber dim 1 at $\underset{[(t)]}{0}$ and 0 everywhere else.