Math 631

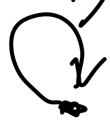
Plan for today:

- categories of (pre)sheaves
- main tool for understanding sheaves: <u>stalks</u>
- sheaves of abelian groups

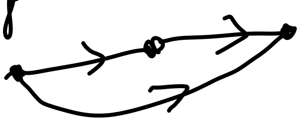
Morphisms of sheaves:

want sheaves to form a category

(Category = objects, morphisms, identity morphisms,

composition of morphisms)



Tempting: try to define a morphism from $\mathcal{F}$ on X to $\mathcal{G}$ on Y

Instead: fix X, consider two sheaves $\mathcal{F}$, $\mathcal{G}$ on X

Want: $\pi: \mathcal{F} \to \mathcal{G}$.

Def: Let X be a top. space. The <u>category of sheaves (of sets) on X</u> is denoted Sets_X.

The objects of Sets_X are sheaves $\mathcal{F}$ on X.

The morphisms $\pi: \mathcal{F} \to \mathcal{G}$ of Sets_X are given by maps (morphisms in Sets)

$\pi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ for every $U \underset{\text{open}}{\subseteq} X$, commuting with restriction:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\pi(U)"] \arrow[d, "r_{UV}"'] \arrow[dr, phantom, "\circ"] & \mathcal{G}(U) \arrow[d, "r_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\pi(V)"] & \mathcal{G}(V) \end{tikzcd} \quad \text{for } V \underset{\text{open}}{\subseteq} U \underset{\text{open}}{\subseteq} X.$$

<u>Note:</u> The sheaf axioms are irrelevant to morphism of sheaves.

We also define $\mathrm{Sets}_X^{\mathrm{pre}}$ as the category of presheaves on X.

<u>Example:</u> X, Y top. spaces, $p \in X$ $\quad \iota_p : \{p\} \hookrightarrow X$.

$\mathcal{F}$ = sheaf on X of cont. functions to Y.

$\mathcal{G}$ = skyscraper sheaf $(\iota_p)_* \underline{Y}$ on X

$\pi : \mathcal{F} \to \mathcal{G}$ morphism given by taking value of function at p.

Easy exercise/definition:

a) If $V \underset{\text{open}}{\subseteq} X$, then $\mathcal{F} \longmapsto \mathcal{F}|_V$ actually gives a <u>functor</u> $|_V : \mathrm{Sets}_X \to \mathrm{Sets}_V$.

b) If $f : X \xrightarrow[\text{cont}]{} Y$, then $\mathcal{F} \longmapsto f_* \mathcal{F}$ " " " " " <u>functor</u> $f_* : \mathrm{Sets}_X \to \mathrm{Sets}_Y$.

(Functor: objects $\mapsto$ objects, morphisms $\mapsto$ morphisms, distribute over composition)

Content: given $\pi : \mathcal{F} \to \mathcal{G}$, have $\pi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$,

$f_* \pi : f_* \mathcal{F} \to f_* \mathcal{G}$.

<u>Def</u>: Let $\mathcal{F}$ be a sheaf on X and $p \in X$. Then the <u>stalk</u> of $\mathcal{F}$ at p is

$$\mathcal{F}_p = \{(U,s) \mid p \in U \underset{\text{open}}{\subseteq} X,\ s \in \mathcal{F}(U)\} / \sim,$$

where $\sim$ is the equiv. relation generated by

$$(U,s) \sim (V, s|_V) \text{ for } p \in V \subseteq U.$$

Elements of $\mathcal{F}_p$ are called <u>germs</u>.

The germ $(U,s) \in \mathcal{F}_p$ is the <u>germ of s at p</u> and is sometimes denoted s_p.

Examples: 1) $\mathcal{F}$ = sheaf on X of cont. functions to Y

$\mathcal{F}_p$ = "cont. functions defined on arbitrarily small open nhoods of p"

2) $\mathcal{F} = (\iota_p)_* S$

$$\mathcal{F}_q = \begin{cases} S & \text{if } q \in \overline{\{p\}} \text{ (top. closure)} \\ \{\cdot\} & \text{else} \end{cases}$$

Exercise/definition: Morphisms of sheaves induce maps on stalks, i.e.

$\pi : \mathcal{F} \to \mathcal{G}$ (so have $\pi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$)

$\rightsquigarrow$

$\pi_p : \mathcal{F}_p \to \mathcal{G}_p$

In other words, $(\ )_p$ is a functor $\mathrm{Sets}_X \to \mathrm{Sets}$.

Lemma: (germs determine sections): Let $\mathcal{F}$ be a sheaf on X, and let $U \underset{\text{open}}{\subseteq} X$. Then the natural map

$$\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p$$
$$s \longmapsto (s_p)_{p \in U}$$

is injective.

Pf: If $(s_1)_p = (s_2)_p$ then $s_1|_{U_p} = s_2|_{U_p}$ for some open nhood $U_p \ni p$.

So the $\{U_p\}$ are an open cover where s_1, s_2 agree, so $s_1 = s_2$ by identity axiom. ▨

Lemma: (morphisms are determined by maps on stalks)

$$\underbrace{\mathrm{Mor}_{\mathrm{Sets}_X}(\mathcal{F},\mathcal{G})}_{\substack{\text{set of morphisms}\\ \text{in category } \mathrm{Sets}_X}} \longrightarrow \prod_{p\in X} \mathrm{Mor}_{\mathrm{Sets}}(\mathcal{F}_p,\mathcal{G}_p)$$

is injective.

Pf: Given $\pi_1,\pi_2 : \mathcal{F}\to\mathcal{G}$ with $(\pi_1)_p = (\pi_2)_p$, want $\pi_1(U) = \pi_2(U)$ for each U.

By the previous lemma, $\pi_1(U)(s) = \pi_2(U)(s)$ (since they have the same germs). ▨

<u>Def</u>: The category of <u>sheaves of abelian groups</u> on X is denoted Ab_X and is identical to $Sets_X$ except that the $\mathcal{F}(U)$ and restriction/sheaf morphism maps between the $\mathcal{F}(U)$ are in $\underbrace{Ab}_{\text{category of abelian groups}}$ instead of Sets.

Can similarly define Ab_X^{pre}, $Rings_X$, $Rings_X^{pre}$, ...

<u>Note</u>: if $\mathcal{F} \in Ab_X$, its stalks $\mathcal{F}_p \in Ab$.

Most of the term: just care about sheaves of rings.

Thursday: discuss theory of Ab_X.

Preview: Ab, $\underbrace{Ab_X^{pre}, Ab_X}_{\text{still have ker, coker, exact sequences.}}$ are very similar categories

("$\underbrace{\text{abelian category}}_{\text{like } Mod_A \text{ cat. of A-modules.}}$")

Last time: • categories Sets_X, Rings_X, Ab_X, Ab_X^{pre}, ...
• stalks $\mathcal{F}_p$

Today: Ab_X behaves like Mod_A (category of A-modules), i.e. is an "abelian category".

What does this mean? $\mathrm{Mor}(A,B) = \mathrm{Hom}(A,B)$ is an abelian group, 0 object, $\oplus$, ker, coker, exact sequences

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First, what about Ab_X^{pre}?

$\mathrm{Ab} = \mathrm{Mod}_{\mathbb{Z}}$ does have all these things and $\mathcal{F} \in \mathrm{Ab}_X^{pre}$ is just a bunch of $\mathcal{F}(U) \in \mathrm{Ab}$ (with some maps)

zero sheaf: $0(U) = 0 \in \mathrm{Ab}$.

zero morphism: $\pi(U) = 0 : \mathcal{F}(U) \to \mathcal{G}(U)$

$(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$

$(\ker^{pre}\pi)(U) = \ker(\pi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$

$(\mathrm{coker}^{pre}\pi)(U) = \ldots$

What about Ab_X?

Potential problem: sheaf axioms aren't obviously compatible with going "open set by open set"

Can check: $\oplus$, ker work fine in Ab_X
(the presheaf construction gives a sheaf)

But coker is bad!

Example: X = top. space ($X = S^1 = \mathbb{R}/\mathbb{Z} = \bigcirc$ circle)

$\mathcal{M}_Y$:= sheaf on X of cont. functions to Y.

Notes: 1) If $\pi: Y \xrightarrow[\text{cont}]{} Z$, then composition gives a sheaf morphism $\mathcal{M}_Y \to \mathcal{M}_Z$

2) If Y is also an ab. group ($\mathbb{R}$) then $\mathcal{M}_Y$ can be viewed as an object in Ab_X.

Consider the maps

$$\mathcal{M}_{\mathbb{Z}} \xrightarrow{f} \mathcal{M}_{\mathbb{R}} \xrightarrow{g} \mathcal{M}_{\mathbb{R}/\mathbb{Z}}$$

coming from $\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$.

Claims: 1) $\operatorname{coker}^{pre} f$ fails gluability
2) $\operatorname{coker}^{pre} g$ fails identify.

Check 2: $\mathcal{M}_{\mathbb{R}} \xrightarrow{g} \mathcal{M}_{\mathbb{R}/\mathbb{Z}}$, sheaves on $X = \mathbb{R}/\mathbb{Z}$

Let $\mathcal{F} = \operatorname{coker}^{pre} g$, so

$$\mathcal{F}(U) = \operatorname{coker}\left(\{U \xrightarrow{\text{cont}} \mathbb{R}\} \to \{U \xrightarrow{\text{cont}} \mathbb{R}/\mathbb{Z}\}\right)$$

for each $U \subseteq X = S^1$

Get: $\mathcal{F}(U) = \begin{cases} 0 & \text{if } U \neq X \\ \mathbb{Z} & \text{if } U = X \end{cases}$, fails identity axiom.

It turns out any morphism in Ab_X has a cokernel, it just isn't necessarily equal to its cokernel as a morphism in Ab_X^{pre}

<u>Def</u> (categorical cokernel):

Let $\mathcal{C}$ be an additive category ($\mathrm{Mor}_{\mathcal{C}}(A,B)$ is an ab. group $(\ldots)$).

The <u>cokernel</u> of a morphism $\pi: A \to B$ in $\mathcal{C}$ is an object $\operatorname{coker}\pi \in \mathcal{C}$ along with $\tau: B \to \operatorname{coker}\pi$ such that $\tau \circ \pi: A \to \operatorname{coker}\pi$ is the zero morphism and $(\operatorname{coker}\pi, \tau)$ is "universal among such pairs", i.e. any $(D, \rho: B \to D)$ satisfying $\rho \circ \pi = 0$ factors uniquely through $(\operatorname{coker}\pi, \tau)$:

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[dr, "0"'] & B \arrow[d, "\tau"] \arrow[drr, "\rho"] \arrow[dr, phantom, "0"] & & \\
& \operatorname{coker}\pi \arrow[rr, dotted, "\exists!"'] & & D
\end{tikzcd}$$

<u>Plan:</u> turn the presheaf $\operatorname{coker}^{pre}\pi$ into a sheaf in a universal way:

__Def__: Let $\mathcal{F}$ be a presheaf on X. The __sheafification__ of $\mathcal{F}$ is a sheaf $\mathcal{F}^{sh}$ on X along with a (presheaf) morphism

$$sh : \mathcal{F} \to \mathcal{F}^{sh} \text{ such that}$$

$(\mathcal{F}^{sh}, sh)$ is universal among such pairs.

For any sheaf $\mathcal{G}$ and morphism $\varphi : \mathcal{F} \to \mathcal{G}$, φ factors uniquely through sh:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^{sh} \arrow[d, dashed, "\exists!"] \\ & \mathcal{G} \end{tikzcd}$$

__Thm__: Sheafifications always exist.

__Pf__: Will describe construction later.

Cor (of $\mathcal{F}^{sh}$ existing):

Let $\pi : \mathcal{F} \to \mathcal{G}$ be a morphism in Ab_X.

Let $\mathcal{G} \xrightarrow{\tau} \mathcal{H} = \operatorname{coker}^{pre}_{\pi}$ be the cokernel presheaf.

Let $\mathcal{H} \xrightarrow{sh} \mathcal{H}^{sh}$ be the sheafification.

Then $\mathcal{G} \xrightarrow{sh \circ \tau} \mathcal{H}^{sh}$ is the cokernel of π in Ab_X.

We want to check the cokernel universal property for $\mathcal{G} \xrightarrow{sh \circ \tau} \mathcal{H}^{sh}$: will $\mathcal{G} \xrightarrow{\rho} \mathcal{I}$ with $\rho \circ \pi = 0$ factor uniquely?

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\pi"] \arrow[dr, "0"] \arrow[dddrr, bend right, "0"'] & \mathcal{G} \arrow[d, "\tau"] \arrow[ddr, "sh \circ \tau"] \arrow[dddr, bend left=60, "\rho"] & \\
 & \mathcal{H} \arrow[dr, "sh"] \arrow[ddr, dashed, "\exists!"'] & \\
 & & \mathcal{H}^{sh} \arrow[d, dashed, "\exists!"] \\
 & & \mathcal{I}
\end{tikzcd}
$$

from univ prop of $\mathcal{H}$.

from univ prop of $\mathcal{H} \to \mathcal{H}^{sh}$

How do we construct $\mathcal{H}^{sh}$? Need to do more with stalks.

$$\mathcal{F} \text{ on } X \rightsquigarrow (\mathcal{F}_p)_{p \in X}$$

Def: Let $\mathcal{F}$ be a (pre)sheaf on X and $U \underset{\text{open}}{\subseteq} X$.

We say $\left(s^{(p)} \in \mathcal{F}_p\right)_{p \in U} \in \prod_{p \in U} \mathcal{F}_p$ are <u>compatible germs</u> if U can be covered by opens V with the property that there is a section $t \in \mathcal{F}(V)$ such that $s^{(p)} = t_p$ for all $p \in V$.

Lemma: Let $\mathcal{F}$ be a presheaf on X. Then $\mathcal{F}$ is a sheaf $\Longleftrightarrow$ for any $U \underset{\text{open}}{\subseteq} X$, the map $\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p$ is injective and has image precisely those with compatible germs.

Construction of $\mathcal{F}^{sh}$: take stalks of presheaf $\mathcal{F}$, and set

$$\mathcal{F}^{sh}(U) = \left\{ (s^{(p)}) \in \prod_{p \in U} \mathcal{F}_p \;\middle|\; \text{compatible germs} \right\}$$

Can check: $\mathcal{F}^{sh}_p \cong \mathcal{F}_p$, compat. germs condition is the same on both sides

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<u>Lemma:</u> "stalks detect isomorphisms"

$\pi : \mathcal{F} \to \mathcal{G}$ is an isom (in Sets_X)

$\iff \pi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isom for all p.

<u>Cor:</u> $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact ($\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Ab}_X$)

$\iff \mathcal{F}_p \to \mathcal{G}_p \to \mathcal{H}_p$ is exact for all p.

Office hours: Mon/Tue 4–5pm, Fri 1–2pm, in EH 3842.
(starting next week)

Homework: first pset posted later today, due 1 week from today.

I'll post both on main website and Gradescope, might take a day or two to fully set up Gradescope (+ Canvas).

Policies:
- welcome to discuss with others, but should write up your own work and acknowledge anyone you worked with.
- can cite results from class, earlier exercises in [FoAG], etc without proof
- you can try using GPT-4 etc if you want, but I don't recomend it. If you do find a LLM useful for solving a problem, please state how you used it.

I will upload and link this file in week 2 of the schedule on the course website.