PROBLEM SET 10 (DUE ON THURSDAY, NOV 16)

(All Exercises are references to the July 31, 2023 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Let $X = \operatorname{Proj} \mathbb{C}[x_0, x_1, x_2, x_3]/(x_0x_3 - x_1x_2)$, a quadric surface in $\mathbb{P}^3_{\mathbb{C}}$. Describe a closed embedding $\iota : \mathbb{P}^1_{\mathbb{C}} \to X$ with the property that the image of ι is not contained in any plane in $\mathbb{P}^3_{\mathbb{C}}$. Then describe a morphism $\pi : X \to \mathbb{P}^1_{\mathbb{C}}$ with the property that $\pi \circ \iota$ is the identity morphism on $\mathbb{P}^1_{\mathbb{C}}$. What do the fibers of π over closed points look like, as closed subsets of $\mathbb{P}^3_{\mathbb{C}}$? (Hint: Veronese embeddings and Exercise 9.3.L (rulings on the quadric surface) might both be helpful here.)

Problem 2. Exercise 10.3.F (blowing up a point in $\mathbb{A}^2_k$ - it might help to note that $\mathbb{A}^2_k \times_k \mathbb{P}^1_k$ is isomorphic to $\operatorname{Proj} k[x, y, u, v]$, where x, y have degree 0 and u, v have degree 1)

Problem 3. Exercise 10.2.J (distinct morphisms remain distinct upon extending the base field)

Problem 4. Describe two morphisms $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ such that the fiber product $X = \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{A}^1_{\mathbb{C}}} \mathbb{A}^1_{\mathbb{C}}$ using these morphisms has exactly two irreducible components and such that the two irreducible components intersect in exactly two points.