PROBLEM SET 12 (DUE ON THURSDAY, DEC 8)

(All Exercises are references to the August 29, 2022 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 12.2.L (most surfaces of degree $d > 3$ have no lines - Vakil gives a detailed outline of how to do this, and you might also find it helpful to read his proof of Bertini's Theorem later (13.4.2), but here is an additional note: if you are unfamiliar with the Grassmannian $\mathbb{G}(1,3)$, you can replace it in this proof with a single affine chart $\mathbb{A}^4$, where the closed point $(x_1, x_2, x_3, x_4) \in \mathbb{A}^4$ corresponds to the line between $[1 : 0 : x_1 : x_2]$ and $[0 : 1 : x_3 : x_4]$ in $\mathbb{P}^3$. You will conclude that "most" degree d surfaces have no lines of this form, and then you can finish by noting that the set of lines in $\mathbb{P}^3$ can be covered by finitely many charts of this type.)

Problem 2. Exercise 12.3.E (closed subvarieties of $\mathbb{P}^n$ intersect when suggested by dimensions - you will want to use some combination of the ideas used in Exercises 12.3.D, 12.3.B, 12.2.F (all of which you can cite without proof if you'd like) as well as the notion of an affine cone from section 9.3.11.)

Problem 3. The *tangent cone* at a point p of a scheme is defined as $\operatorname{Spec} \bigoplus_{i \geq 0} \mathfrak{m}_p^i / \mathfrak{m}_p^{i+1}$, where the direct sum is given a ring structure in the natural way. Let $X = \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^2)$ (two transverse lines) and $Y = \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^2 - x^3)$ (a nodal cubic curve). Show that X and Y have isomorphic tangent cones at the origin. (This is one way of making sense of the statement that these two curve singularities are the "same type" - a simple node.)

Problem 4. Do Exercise 13.3.N (assuming Exercise 13.3.M). Then show that the tangent cone of $\operatorname{Spec} \mathbb{Z}[5i]$ at the point $[(5, 5i)]$ is isomorphic to the tangent cone of $\operatorname{Spec} \mathbb{F}_5[x,y]/(xy)$ at the origin (the point $[(x,y)]$). (In other words, the singularity at $[(5, 5i)] \in \operatorname{Spec} \mathbb{Z}[5i]$ can also be thought of as a simple node.)

Problem 5. Suppose that X and Y are closed subvarieties of $\mathbb{P}^n_k$ of pure dimension d and $n - d$ respectively. Suppose that p is an isolated point (i.e. a connected component) in the intersection of X and Y, and suppose that X is singular at p. Show that the scheme-theoretic intersection $X \cap Y$ is not reduced at p. (Hint: Exercise 13.1.C might be helpful and can be assumed without proof.)