Last time: line bundles on a scheme:

- L is an $\mathcal{O}_X$-module that is $\underbrace{\text{locally trivial}}_{L|_U \cong \mathcal{O}_U}$ on opens U covering X

- motivation: sections of a morphism $X' \to X$ that looks like $U \times \mathbb{A}^1 \to U$ on opens U that cover X (glued via scaling automorphisms $\mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$ on fibers)
$t \mapsto ct$

- <u>Example</u>: $L = \mathcal{O}_{\mathbb{P}^n_k}(d)$, $(d \in \mathbb{Z})$ line bundle on $\mathbb{P}^n_k$ with sections

$$L(D(f)) = \left\{ \text{rational functions } \frac{g}{h} \in k(x_0, \dots, x_n) \;\middle|\; \begin{array}{l} g, h \text{ homog.} \\ h = f^m \text{ for } m \geq 0 \\ \deg g - \deg h = d \end{array} \right\}$$

$L(\mathbb{P}^n_k) = k$-vector space with basis
$x_0^d, x_0^{d-1} x_1, \dots, x_n^d$.

Pullbacks of line bundles:

idea: if $X' \to X$ has "fibers that are 1-dim vector spaces", and $Y \to X$ is a morphism, then expect

$$Y' := Y \times_X X' \to Y \text{ also has such fibers.}$$

"sections of $X' \to X$" $\overset{\text{pullback}}{\rightsquigarrow}$ "sections of $Y' \to Y$".

Def: Suppose $\mathcal{F}$ is a qcoh sheaf on X and $\pi : Y \to X$ is a morphism. Then the <u>pullback sheaf</u>, denoted $\pi^* \mathcal{F}$, is a qcoh sheaf on Y determined by

$$\underbrace{(\pi^* \mathcal{F})(\operatorname{Spec} A)}_{A\text{-module}} = \underbrace{\mathcal{F}(\operatorname{Spec} B)}_{B\text{-module}} \otimes_B A \quad \text{for any}$$

$s \otimes 1$

affine opens $\operatorname{Spec} A \subseteq Y$, $\operatorname{Spec} B \subseteq X$ with $\pi(\operatorname{Spec} A) \subseteq \operatorname{Spec} B$.

Can check: $\pi^* \mathcal{O}_X \cong \mathcal{O}_Y$, so pullback of a line bundle is a line bundle.

Thm 1 (16.4.1 in Vakil): Let X be an A-scheme. Let $n \geq 1$. Then there is a bijection

$$\{\text{morphisms } \pi : X \to \mathbb{P}^n_A\} \xrightarrow{\sim} \left\{ (L, s_0, \dots, s_n) \;\middle|\; \begin{array}{l} L \text{ is a line bundle on } X, \\ s_i \in L(X) \text{ have no} \\ \text{common zeroes} \end{array} \right\} \Big/ \text{isom.}$$

$$\pi \longmapsto (\pi^* \mathcal{O}_{\mathbb{P}^n}(1), \pi^* x_0, \dots, \pi^* x_n)$$

Notes: 1) pullback of sections is really $\pi^* x_i = "x_i \otimes 1"$.

2) vanishing of a section s of a line bundle L at a point p can be thought of in two ways:

a) Restrict to $U \ni p$ where $L|_U \cong \mathcal{O}_U$ and then check vanishing of s as an element of $\mathcal{O}_U(U)$.

b) Check whether s maps to 0 in

$$L_p \otimes_{\mathcal{O}_{X,p}} k_p \underset{\uparrow}{\cong} k_p$$

as k_p-vector space.

3) Knowledge of line bundles on X tells you a lot about X,

e.g. Prop: Every line bundle on $\mathbb{P}^n_k$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \in \mathbb{Z}$.

Cor: All morphisms $\mathbb{P}^m_k \to \mathbb{P}^n_k$ are induced by maps of graded rings $[x_0 : \cdots : x_m] \mapsto \boxed{[f_0 : \cdots : f_n]}$ homog. of deg d

$y_i \mapsto f_i(x_0, \ldots, x_m)$

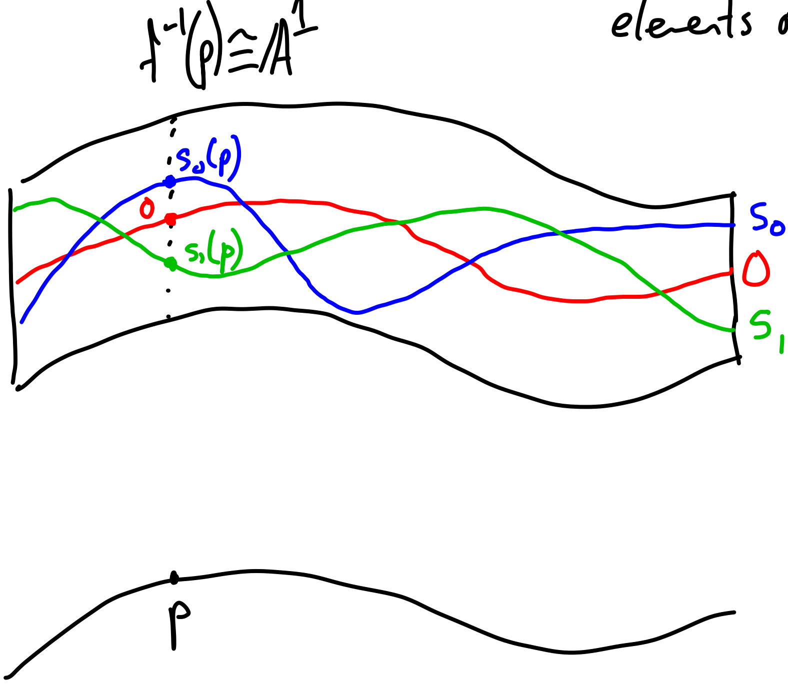
$\operatorname{Proj} k[x_0, \ldots, x_m] \to \operatorname{Proj} k[y_0, \ldots, y_n]$

Sketch of pf of Thm 1: define the inverse to the proposed bijection:

Given $(L, s_0, \dots, s_n)$, define $\pi : X \to \mathbb{P}^n$ as follows.

Informally: $\pi(p) = [s_0(p) : \cdots : s_n(p)]$

elements of $L_p \otimes_{\mathcal{O}_{X,p}} k_p \cong k_p$



$$\begin{array}{ccc} X' & & \\ \big\downarrow f & & \\ X & \xrightarrow{\pi} & \mathbb{P}^n_A \\ p & \longmapsto & [s_0(p) : \cdots : s_n(p)] \end{array}$$

$(X \longrightarrow \mathbb{P}^1$

$p \longmapsto$ " $\dfrac{s_0(p)}{s_1(p)}$ "

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formally: on $U \subseteq X$ where $L \cong \mathcal{O}_X$, $s_0, \ldots, s_n$ corresp. to functions $f_0, \ldots, f_n \in \mathcal{O}_X(U)$, glue together morphisms

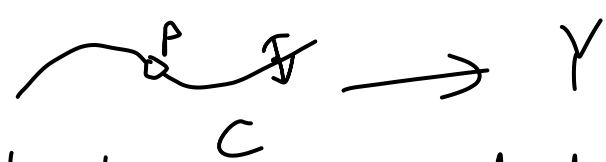
$$D(f_i) \longrightarrow D(x_i) \subseteq \mathbb{P}^n_A$$

$$\frac{f_j}{f_i} \longleftarrow\!\shortmid \frac{x_j}{x_i} .$$

Thm 2 (Vakil: 16.5.1, "Curve-to-Projective Extension Thm")

A = ring
C = Noetherian A-scheme that is a curve
$p \in C$ is a regular closed point
Y = projective A-scheme
$\pi : C \setminus \{p\} \to Y$ is a morphism (of A-schemes)
Then π can be (uniquely) extended to a morphism $C \to Y$.

Sketch of pf:



Step 1: Can assume C is reduced by shrinking C (C is reduced at p)

Step 2: By regularity, $\mathcal{O}_{C,p}$ is a DVR. Choose an element $t \in \mathcal{O}_{C,p}$ with $v_p(t) = 1$ and shrink C again until we can realize $t \in \mathcal{O}_C(C)$ with $V(t) = \{p\}$.

Step 3: argue that it suffices to consider the case $Y = \mathbb{P}^n_A$.
(compose π with $Y \hookrightarrow \mathbb{P}^n_A$, then extend to get a morphism $C \to \mathbb{P}^n_A$, then factor through Y using the fact that C is reduced).

Step 4: By Thm 1, $\pi : C \setminus \{p\} \to \mathbb{P}^n_A$ corresponds to data $(L, s_0, \ldots, s_n)$ on $C \setminus \{p\}$.

Shrink C again so that $L \cong \mathcal{O}_{C \setminus \{p\}}$ and

$s_0, \ldots, s_n \in \mathcal{O}_C(C \setminus \{p\})$.

View $s_0, \ldots, s_n$ as elements of $K(C)$ = field of fractions of $\underbrace{\mathcal{O}_{C,p}}_{\text{DVR}}$.

The DVR valuation v_p extends to $K(C)$, so can define

$m = \min(v_p(s_i))$. Then

$$(\mathcal{O}_C, t^{-m} s_0, t^{-m} s_1, \ldots, t^{-m} s_n)$$

defines the desired extension $C \to \mathbb{P}^n_A$.

$$\left(\begin{array}{l} [t : t^3 + t^2 : t^{-2} + 1] : \mathbb{A}^1_k - \{0\} \to \mathbb{P}^2_k \\ \quad \rightsquigarrow [t^3 : t^5 + t^4 : 1 + t^2] : \mathbb{A}^1_k \to \mathbb{P}^2_k \end{array} \right)$$

($m \in \mathbb{Z}$ was a choice of how to extend the data $(L, s_0, \ldots, s_n)$ over the point p, and one choice works.) $\blacksquare$

One application:

Thm 3: (see Thm 17.4.1–17.4.3 in Vakil).

Let k be a field and let C be an irred curve that is a k-variety. Then C has a unique regular projective birational model, i.e. a rational map of k-schemes

$\pi : C \dashrightarrow C'$ which is an isomorphism on some open dense subsets, where C' is a regular projective curve. ("unique" means up to composition with an isomorphism $C' \to C''$).

Moreover, if C is regular then π is an open embedding.

($C \dashrightarrow C \setminus \{\text{singular points}\} \underset{\text{open embedding}}{\hookrightarrow} C'$.

"curves have unique regular proj. models"

Uniqueness fails in higher dimension, e.g.

$\mathbb{P}^2 \not\cong \mathbb{P}^1 \times \mathbb{P}^1$.

__Sketch of pf:__

__Uniqueness:__ this is equivalent to saying that any birational equivalence $C' \dashrightarrow C''$ of regular proj. curves is an isomorphism, which follows from applying Thm 2 in both directions.

__Existence:__ $C \rightsquigarrow$ affine open $C' \rightsquigarrow$ closed subscheme $C' \hookrightarrow \mathbb{A}^n_k \rightsquigarrow$ closure $\overline{C'} \hookrightarrow \mathbb{P}^n_k$.

This is proj but not necessarily regular, so take normalization $\widetilde{\overline{C'}}$ of $\overline{C'}$.

__Fact:__ normalization of proj. scheme is proj.

$$\begin{tikzcd} & \widetilde{\overline{C'}} \arrow[dd, "\exists \text{ by uniqueness}", "\wr"'] \\ C \arrow[ur, dashed] \arrow[dr, dashed] & \\ & \widetilde{\overline{C''}} \end{tikzcd}$$