Last time: a regular local ring of dim 1 is called a DVR, many nice features.

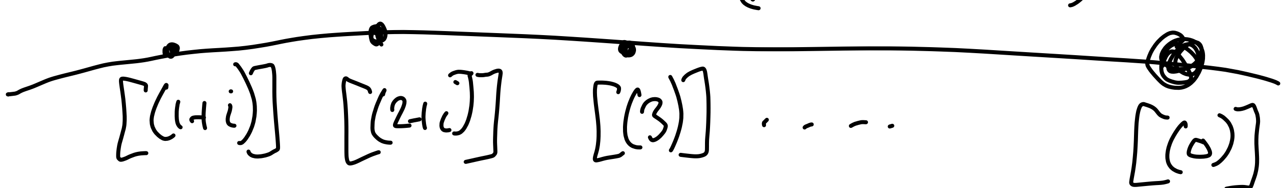
Remarks:

1) Noetherian local rings of dim 1 that are integrally closed in their field of fractions are DVRs.

Consequence: Suppose $A$ is a Noetherian domain of $\underbrace{\text{dim } 1}$ that is integrally closed in its field of fractions ("Dedekind domain"). Then $\operatorname{Spec} A$ is a regular curve

Example: $A$ is the ring of integers of a number field, e.g. $\mathbb{Z}[i]$.

So $\operatorname{Spec} \mathbb{Z}[i]$ is a regular $\underbrace{\text{curve}}$. (pure of dim. 1)




tangent spaces: $\mathbb{F}_2$ $\mathbb{F}_5$ $\mathbb{F}_9$ . . . . .

2) Another consequence: if $X$ is a $\underbrace{\text{normal}}$ loc. Noetherian scheme, then $X$ is "regular in codim 1," i.e. $\mathcal{O}_{X,p}$ is a DVR for $\operatorname{codim}_X p = 1$.

(normal $\iff$ regular for curves)

3) Let $(A, m)$ be a DVR. Then there is a <u>valuation</u>

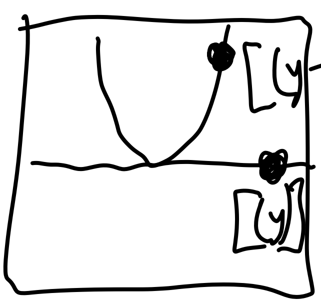
$v: A \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ defined by

$v(f) = n \iff f \in m^n, f \notin m^{n+1}$.

<u>Consequence:</u> If $X$ is regular at a point $p$ of codim 1 and $f \in \mathcal{O}_X(U)$ for open $U \ni p$, then we can define the <u>order of vanishing of $f$ at $p$</u> by $v_p(f)$

($v_p$ = valuation on $(\mathcal{O}_{X,p}, m_p)$).

<u>Example:</u> $X = \mathbb{A}^2_k$, $f = (y-x^2)^2 y^3 \in \mathcal{O}_X(X)$.




$f$ vanishes to order 2 at $[(y-x^2)]$

(or "along $V(y-x^2)$")

"$f$ has a double zero at $[(y-x^2)]$"

Feature of DVR: if $v_p(f) \geq v_p(g)$, then

"$\frac{f}{g}$ is defined at $p$" (in $\mathcal{O}_{X,p}$, there exists $h \in \mathcal{O}_{X,p}$ with $f = gh$)

(Compare with $\frac{x}{y} \in K(\mathbb{A}^2_k)$, which is not defined at the origin).

Goal for remainder of term: discuss three theorems dealing with projective schemes and/or regular curves.

There will be a rough progression

Thm 1 $\rightsquigarrow$ Thm 2 $\rightsquigarrow$ Thm 3.

Today: state Thm 2, start building concepts/defs needed for Thm 1.

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Thm 2 (Vakil: 16.5.1, "Curve-to-Projective Extension Thm")

Let $A$ be a ring. Suppose:

$C$ is a Noetherian $A$-scheme that is a curve (pure of dim. 1)

$p \in C$ is a regular closed point.

$Y$ is a projective $A$-scheme

$\pi : C \setminus \{p\} \longrightarrow Y$ is a morphism (of $A$-schemes).

Then $\pi$ can be extended (uniquely) to a morphism $C \longrightarrow Y$.

Cor (as promised): Let $C$ be a regular irred curve that is an $A$-scheme. Then any rational map $C \dashrightarrow \mathbb{A}^1_A$ extends uniquely to a morphism $C \longrightarrow \mathbb{P}^1_A$.

Note: This theorem is true more generally for $Y$ a proper $A$-scheme. (recall: proj $\Rightarrow$ proper)

instead of proj.

(Could prove curve-to-proper extension theorem and also proj $\Rightarrow$ proper)

(We'll use a more geometric approach.)

<u>Recall</u>: an $\mathcal{O}_X$-module is a sheaf of ab. groups $\mathcal{F}$ on $X$ s.t. $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.

<u>Def</u>: A <u>quasicoherent sheaf</u> on a scheme $X$ is an $\mathcal{O}_X$-module $\mathcal{F}$ s.t.

$$\mathcal{F}(\operatorname{Spec} A)\left[\tfrac{1}{f}\right] \cong \mathcal{F}(D_{\operatorname{Spec} A}(f))$$

for affine open $\operatorname{Spec} A \subseteq X$ and $f \in A$.

(Example: $\mathcal{O}_X$ is qcoh)

There is a correspondence

$$\{\text{qcoh sheaf } \mathcal{F} \text{ on } \operatorname{Spec} A\} \overset{\sim}{\longleftrightarrow} \{A\text{-module } M\}$$

$$\mathcal{F} \longmapsto \mathcal{F}(\operatorname{Spec} A)$$

"qcoh sheaf = something formed by gluing modules over different rings"

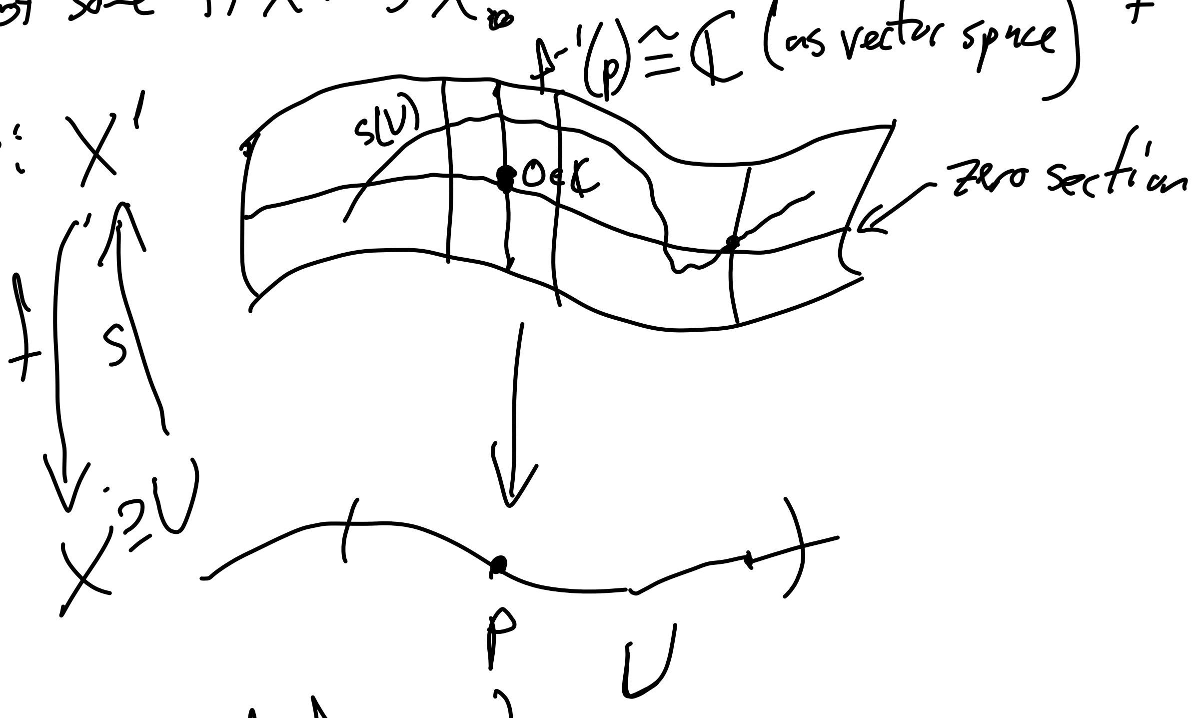
**Def**: A line bundle (or invertible sheaf) on a scheme $X$ is a qcoh sheaf $L$ s.t. $X$ can be covered by open sets $U$ with $L|_U \cong \mathcal{O}_U$ (as $\mathcal{O}_U$-modules)

("locally trivial")

**Geom intuition**: recall that sheaves on $X$ "usually" can be thought of as spaces of sections $s: X \to X'$ of some $f: X' \to X$.

$$\begin{tikzcd} X \arrow[r, "s"] & X' \arrow[l, bend left, "f"] \end{tikzcd}$$

line bundle: $X'$




$L(U) = \{\text{sections of } f \text{ over } U\}$

If $\underbrace{f^{-1}(U) \cong U \times \mathbb{C}}_{\text{top. notion of "locally trivial"}}$, then $\underline{L(U)} \cong \{\text{cont. maps to } \mathbb{C}\}$

$\cong \underline{\mathcal{O}_U(U)}$.

(Remark: a general qcoh. sheaf is like a "line bundle whose fibers are vector spaces of varying dim")

## $3^*$ examples of nontrivial line bundles on $\mathbb{P}^n_k$

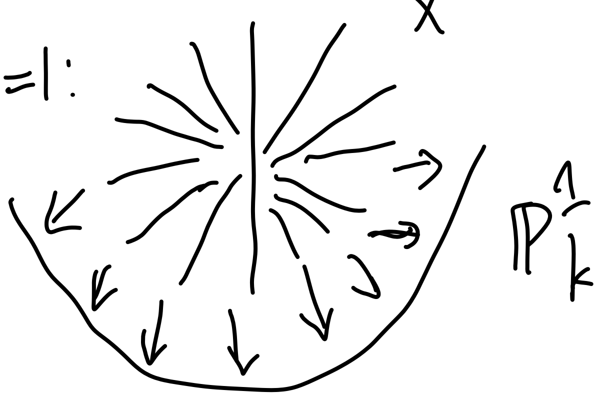
1) Let $X' = \{(\ell, p) \in \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k \mid p \text{ is on the line } \ell\}$

$(a_0, \ldots, a_{n})$ proportional to $[x_0 : x_1 : \cdots : x_n]$ $(a_i x_j - a_j x_i = 0)$

and let $L_1$ be the sheaf of $\underbrace{\text{sections}}_{\text{as } k\text{-scheme morphisms}}$ of the projection $f: X' \to \mathbb{P}^n_k$

(This defines a line bundle because $f^{-1}(D(x_i)) \cong D(x_i) \times \mathbb{A}^1_k$).

$n=1$:




classical $\mathbb{R}$-picture is a Möbius strip.

2) Let $L_2$ be the ideal sheaf $\mathcal{I}_{V(x_0)}$ in $\mathbb{P}^n_k$.

This is a line bundle because on

$D(x_i)$ it corresponds to the <u>principal ideal</u>

$\left(\frac{x_0}{x_i}\right)$, which is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(D(x_i))$

as a module.

[Special case of: if $Z$ is a closed subscheme locally cut out by a single function, then $\mathcal{I}_Z$ is a line bundle.)]

Fact: $L_1 \cong L_2$

3) Let $S_\bullet = k[x_0, \dots, x_n]$

Let $L_3$ be the $\mathcal{O}_{\mathbb{P}^n_k}$-module defined on distinguished opens:

$$\boxed{L_3(D(f)) := (S_\bullet[\tfrac{1}{f}])_1}$$

(module over $(S_\bullet[\frac{1}{f}])_0 = \mathcal{O}_{\mathbb{P}^n_k}(D(f))$.)

Can check: $L_3$ is a line bundle (trivial on $D(x_i)$)

<u>Examples</u>: $x_0, \dots, x_n$ are global sections of $L_3$.

$\frac{x_0 x_1}{x_2}$ is a section of $L_3$ over $D(x_2)$.

More generally: if $S_\bullet$ is gen in degree 1, we define line bundles

$\mathcal{O}_{\operatorname{Proj} S}(d)$ for $d \in \mathbb{Z}$ by

$$(\mathcal{O}_{\operatorname{Proj} S}(d))(D(f)) := (S_\bullet[\tfrac{1}{f}])_d.$$

$\boxed{L_3 = \mathcal{O}_{\mathbb{P}^n_k}(1)}$, turns out $L_1 \cong L_2 \cong \mathcal{O}_{\mathbb{P}^n_k}(-1)$,