Two weeks ago: dimension: many perspectives:

- max length of chain of irred. closed subsets
- transcendence degree of $K(X)/k$ if X is a k-variety
- $\sup_{p \in X} \operatorname{codim}_X p$, where $\operatorname{codim}_X p$ is either:
  - max length of chain of prime ideals in $\mathcal{O}_{X,p}$
  - min number of functions needed to cut out $\overline{\{p\}}$ loc. inside X (assuming Noetherianity).

<u>Today</u>: a fifth perspective, except it isn't always equal to the others.

<u>Tangent spaces</u>:

With manifolds: $T_{X,p}$ = vector space of rank $\dim X$.

A map $\begin{array}{ccc} X & \longrightarrow & Y \\ p & \longmapsto & q \end{array}$ can be linearized to get a map $T_{X,p} \longrightarrow T_{Y,q}$.

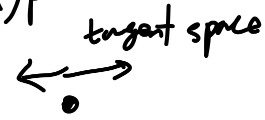
3 motivations for what we will do:

1) tangent vectors $\longleftrightarrow$ derivations on germs at p
2) linearization: $X = \mathbb{A}^n_k$, $p = [(x_1, \ldots, x_n)]$
   $f \in k[x_1, \ldots, x_n] \rightsquigarrow$ "remove terms of deg ≥ 2" $\rightsquigarrow$ mod out by $(x_1, \ldots, x_n)^2$.

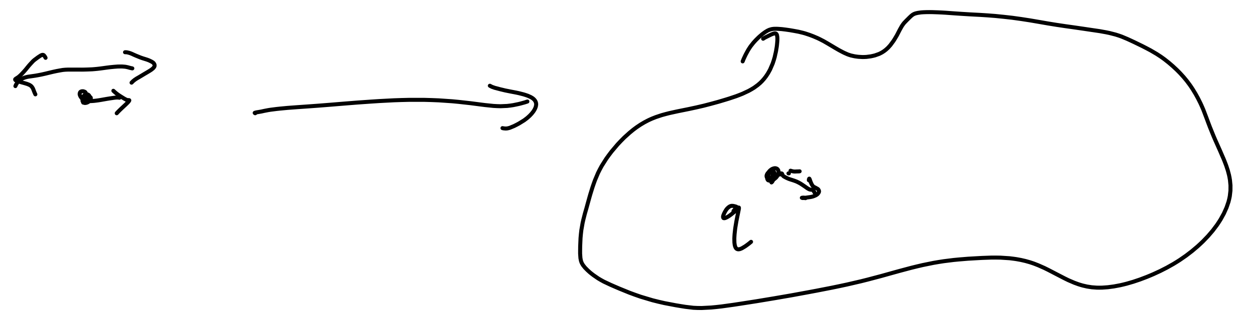
Vakil.

3) $X = \operatorname{Spec} k[\varepsilon]/\varepsilon^2 = \{p\}$

Assume $T_{X,p} \cong k$, i.e.

$X =$ 

So morphisms $X \longrightarrow Y$, $p \longmapsto q$ should corresp. to a choice of tangent vector in $T_{Y,q}$



Get the following:

<u>Def</u>: Let p be a point in a scheme X. The (Zariski) <u>cotangent space</u> to X at p is

$$T^\vee_{X,p} := m_p/m_p^2,$$ where m_p is the max. ideal in $\mathcal{O}_{X,p}$.

This is a vector space over the residue field $k_p = \mathcal{O}_{X,p}/m_p$. $\left(m_p/m_p^2 \cong m_p \otimes_{\mathcal{O}_{X,p}} k_p\right)$

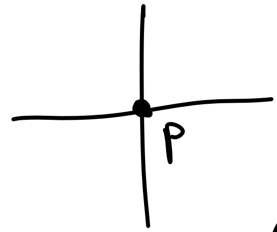
The (Zariski) <u>tangent space</u> is

$$T_{X,p} := (m_p/m_p^2)^\vee$$ (dual vector space).

(finite-dim. if X is loc. Noetherian).

<u>Example:</u> $X = \mathbb{A}^n_k$, $p = [(x_1, \dots, x_n)]$:

$$T^\vee_{X,p} \cong (x_1, \dots, x_n)/(x_1, \dots, x_n)^2$$
$$\cong k^n, \text{ basis } x_1, \dots, x_n.$$

Example (pset): $X =$ 

$T_{X,p} \cong k^2$.

$\quad\ \,\| $
$V(xy) \subseteq \mathbb{A}^2_k$

(even though $\operatorname{codim}_X p = 1$).

[$T_{X,p}$ = subspace of $T_{\mathbb{A}^2_k, p}$ cut out by xy] but $xy \in m_p^2$, so just get $T_{\mathbb{A}^2_k, p} \cong k^2$.

<u>Nice features:</u>

1) A morphism $\pi: X \to Y$ induces a map $T_{X,p} \to T_{Y,q}$ if $\pi(p) = q$ and $k_p = k_q$.

2) If $f \in \mathcal{O}_X(X)$ and $f(p) = 0$ ($\Leftrightarrow f \in m_p$), then the map $T_{V(f),p} \to T_{X,p}$ is an injection and the image is precisely the subspace of $T_{X,p} = (m_p/m_p^2)^\vee$ where f (mod m_p^2) vanishes.

Cor of 2): If $f \in m_p^2$, then $T_{V(f),p} \cong T_{X,p}$.
If $f \notin m_p^2$, then $\dim T_{V(f),p} = \dim T_{X,p} - 1$.

Thm: Suppose (A, m) is a Noetherian local ring. Then
$$\dim_{A/m}(m/m^2) \geq \dim A$$

Pf: Comm. alg (see: Nakayama's Lemma, sec. 7.2.)

"dim of tangent space $\geq$ dim of scheme (near the point)".

Def: A regular local ring is a Noetherian local ring (A, m) with $\dim_{A/m} m/m^2 = \dim A$. A loc. Noetherian scheme X is regular (or nonsingular) at p if $\mathcal{O}_{X,p}$ is a regular local ring, and singular at p otherwise. X is regular if X is regular at every point.

Example: $\mathbb{A}^1_k$ is regular; hence $\mathbb{P}^1_k$ is regular.

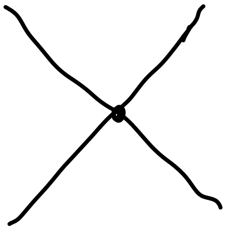
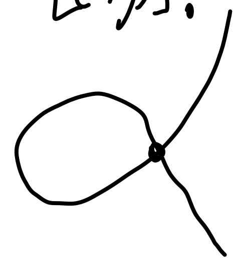
A little more work: $\mathbb{A}^2_k$ is regular.

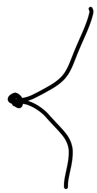
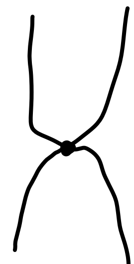
A lot more work: $\mathbb{A}^n_k$ is regular.

Hard fact: if A is a regular local ring and $\mathfrak{p} \subset A$ is prime, then $A_{\mathfrak{p}}$ is a regular local ring.

Consequence: X is regular $\Longleftrightarrow$ X is regular at every closed point.
(X q.c.)

Plane curve singularities: if $f \in (x,y)^2$, then $V(f) \subset \mathbb{A}^2_k$ will have $\dim_k T_{V(f),p} = 2$ for $p = [(x,y)]$.

$V(y^2-x^2)$  $V(y^2-x^2-x^3)$ 

$V(y^2-x^3)$  $V(y^2-x^4)$ 

The notion of tangent cone gives an interesting invariant of singularities (see pset)

(most basic invariant of $p \in X$ is just $\mathcal{O}_{X,p}$).

Example: At which closed points is

$$X = \mathrm{Proj}\ k[x_1, \dots, x_n]/(f) \text{ singular?}$$

(Assume $k = \bar{k}$ for simplicity)

X is singular at $p = [a_0 : \dots : a_n]$

$\iff f_p \in \text{"}m_p^2\text{"} = (x_0 - a_0, \dots, x_n - a_n)^2$

$\iff \dfrac{\partial f}{\partial x_i}(a_0, \dots, a_n) = 0$ for $i = 0, \dots, n$.

$$\iff p \in \underbrace{V\left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)}_{\text{intersection of } n+1 \text{ hypersurfaces in } \mathbb{P}^n}$$

– expect empty for "most" f.

making this precise: consider

special case of Bertini's theorems (sec. 12.4)

$$Z \subset \underbrace{\mathbb{P}^N_k}_{\substack{\text{param} \\ \text{homog. } f \text{ of deg } d}} \times_{\mathrm{Spec}\, k} \underbrace{\mathbb{P}^n_k}_{p}, \quad Z = \Big\{ (f, [x_0 : \dots : x_n]) \,\Big|\, \overbrace{\frac{\partial f}{\partial x_0} = \dots = \frac{\partial f}{\partial x_n} = 0}^{n+1} \Big\}$$

$$= \{ (f, p) \mid V(f) \text{ is singular at } p \}.$$

Claim: $pr_1 : Z \to \mathbb{P}^N_k$ has empty fibers over an open dense set

("most hypersurfaces are regular")

Pf of claim: $\dim Z = N - 1$ (since fibers of $pr_2 : Z \to \mathbb{P}^n_k$ are dim $N - n - 1$.)

Vakil defines (12.2?) the notion of a k-variety being "smooth over k"

We won't use this; if k is a perfect field then smooth $\iff$ regular.

Regularity in low dimension:

Dimension 0:

- A regular local ring of dim 0 is a field. ($m = m^2 \Rightarrow m = 0$)
- Noetherian schemes of dim 0 are disjoint unions of points.

So regular schemes of dim 0 are disjoint unions

$$\operatorname{Spec} k_1 \sqcup \operatorname{Spec} k_2 \sqcup \cdots \text{ for fields } k_1, k_2, \ldots.$$

So for a finite k-scheme, regular $\iff$ reduced.

(can check more generally that regular $\Rightarrow$ reduced for any scheme.)

Dim 1:

A regular local ring of dim 1 is called a <u>discrete valuation ring</u> (DVR).

<u>Example:</u> $k[t]_{(t)}$, $\mathbb{Z}_{(p)}$, $k[x,y]_{(f)}$ (f irred. poly).

$\mathcal{O}_{X,p}$, p codim 1 in X.

Thm: Suppose (A, m) is a Noetherian local ring of dim 1.
Then the following are equivalent:

(a) (A, m) is regular (i.e. $\dim_{A/m} m/m^2 = 1$).

(b) m is principal ($m = (f)$ for some $f \in A$).

(c) The ideals of A are $0, m, m^2, m^3, \ldots$.

(d) A is a principal ideal domain (all ideals are principal).

(e) A is a domain and is integrally closed in $A_{(0)}$.

(f) There exists a "uniformizer" $f \in A$ s.t. any nonzero $x \in A$ can be (uniquely) written $x = f^n u$ with $n \geq 0$, $u \in A^*$.

("discrete valuation" $v(x) := n$, f can be any generator of m)