Last time: $\dim X$ via chains (of irred. closed or primes), finite morphisms preserve dimension

Warning: today will have a lot of commutative alg. results used without proof (Vakil has proofs, or standard refs in comm. alg, e.g. Atiyah–Macdonald)

Our focus: geometric interpretation/consequences/intuition.

Vague goal: build $\underbrace{\text{several}}_{\sim 5}$ different ways of defining/computing/thinking about dimension.

Q: How to see that $\dim \mathbb{A}^n_k = \dim \mathbb{P}^n_k = n$?

Intuition for dim of irred. k-variety:

Should be lots of chains of maximum length:
point $\subset$ "curve" $\subset$ "surface" $\subset \cdots$, so passing from X to a dense open subset shouldn't affect dimension.

Thus if X is an irred. k-variety, expect that dim is a property of the f.g. field extension $K(X)/k$.

So want an invariant of f.g. field extensions that gives n for $k(t_1, \ldots, t_n)/k$.

Def: A field extension K/k has transcendence degree n if there is an intermediate extension K' s.t. $K' \cong k(t_1, \ldots, t_n)$ and K/K' is algebraic.

Thm: Let X be an irred. k-variety. Then
$$\dim X = \text{tr. deg. } K(X)/k.$$

Cor: $\dim \mathbb{A}^n_k = \dim \mathbb{P}^n_k = n$.

Cor: $\dim \operatorname{Spec} k[t_1, \ldots, t_n]/(f) = n-1$ for irred. f
(tr. basis $t_1, \ldots, t_{n-1}$ if f includes t_n)

<u>Pf:</u> Induct on n, in 2 steps;

1) reduce from X with tr. deg. $K(X)/k = n$ to $\mathbb{A}^n_k$:

<u>Noether Normalization Lemma:</u>
Let X be an irred affine k-variety s.t. tr. deg. $K(X)/k = n$. Then there is a finite surjection $X \longrightarrow \mathbb{A}^n_k$.

2) <u>reduce from $\mathbb{A}^n_k$ to some X with tr. deg $K(X)/k = n-1$:</u>

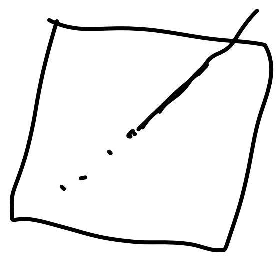
Let $A = k[x_1, \ldots, x_n]$. The chain
$0 \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \ldots, x_n)$ means $\dim A \geq n$.

In the other direction, suppose we have a chain
$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d \subset A$. Let $f \in \mathfrak{p}_1$ be an irred poly. Then
$\mathfrak{p}_1/(f) \subset \mathfrak{p}_2/(f) \subset \mathfrak{p}_3/(f) \subset \cdots \subset \mathfrak{p}_d/(f) \subset A/(f)$
is a chain of primes of length $d-1$. By inductive hypothesis for $\operatorname{Spec} A/(f)$, then $d-1 \leq n-1$.

$\blacksquare$

Codimension:

A general scheme might have multiple irred. components of different dimensions.



overall dim is 2

Def: A top. space X has pure dimension n if every irred. component of X has dim n. A curve is a scheme of pure dimension 1.

Def: Let Y be an irred. subset of a top. space X. The codimension $\operatorname{codim}_X Y$ is the supremum of lengths of chains of irred. closed sets

$$\overline{Y} = Z_0 \subset Z_1 \subset \cdots \subset Z_n \subseteq X.$$

Easy fact: $\operatorname{codim}_X p = \underbrace{\dim \mathcal{O}_{X,p}}$ for any point $p \in X$ where X is a scheme.

Hard fact: Suppose X is a pure-dimensional k-variety. Then $\operatorname{codim}_X Y = \dim X - \dim Y$ for all closed irred. Y. ($\Leftrightarrow$ any Y fits in some maximal chain for X.)

Pf in 11.2.11

Note: $p \in X$ has codim $0 \iff \overline{\{p\}}$ is an irred. component of $X \iff$ (on affines) $\mathfrak{p}$ is a minimal prime ideal of A $\iff$ "$\mathfrak{p}$ has height 0".

One more comm. alg. statement:

Krull's Principal Ideal Theorem:

Suppose A is a Noetherian ring and $f \in A$. Let $\mathfrak{p} \subset A$ be a prime ideal minimal among those containing f. Then $\dim A_{\mathfrak{p}} \leq 1$, i.e. there does not exist a chain of primes $q_0 \subset q_1 \subset \mathfrak{p}$.

alg. statement

Geom. version: ("hypersurfaces are codim 1")

Suppose X is a loc. Noetherian scheme and $f \in \mathcal{O}_X(X)$. Let Y be an irred. component of $V(f)$. Then $\operatorname{codim}_X Y \leq 1$.

Example: If X is a k-variety of pure dimension n and $f \in \mathcal{O}_X(X)$ doesn't vanish identically on any irred. component of X, then $V(f)$ is of pure dim $n-1$.

Nice consequence of last two theorems:

Suppose $X, Y \subseteq \mathbb{P}^n_k$ are closed subschemes of dims d, e with $d+e \geq n$. Then $X \cap Y \neq \emptyset$.

(pset).

Codim of a point is a nice local way of thinking about dimension, e.g.

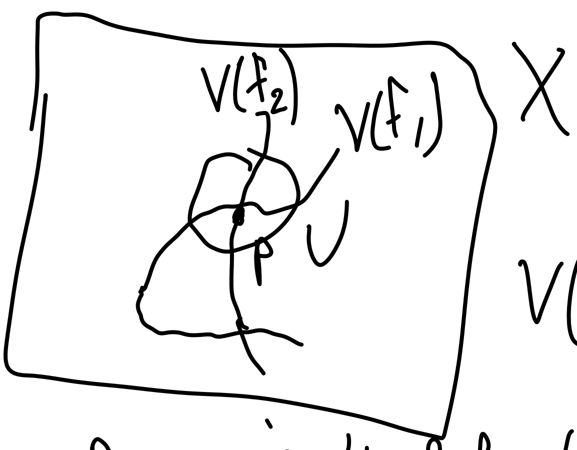
$$\dim X = \sup_{p \in X} (\operatorname{codim}_X p) = \sup_{p \in X} (\underbrace{\dim \mathcal{O}_{X,p}}_{\text{dim of a local ring.}}).$$

Dimension of local ring is especially nice in the sense of having more alt. defs.

<u>Thm (alt. def)</u>: Let (A, m) be a Noetherian local ring. Let $d \geq 0$ be the minimum value such that there exist $f_1, \ldots, f_d \in A$ with $V(f_1, \ldots, f_d) = \{[m]\}$, i.e. $\sqrt{(f_1, \ldots, f_d)} = m$. Then $d = \dim A$.

$$V(f_1, \ldots, f_d) \underset{\operatorname{Spec} A}{\subseteq}$$

<u>Pf</u>: comm. alg, though $d \leq \dim A$ is relatively straightforward.



$A = \mathcal{O}_{X,p}$, m = functions vanishing at p.

$V(f_1) \cap V(f_2) \cap \cdots \cap V(f_d) = \{p\}$ "near p"

$\operatorname{codim}_X p$ = min. # of functions needed to cut out $p \in U$.

Def: $f_1, \dots, f_{\dim A}$ as in this theorem is a <u>system of parameters</u> for (A, m). If X is a loc. Noetherian schee, $f_1, \dots, f_{\operatorname{codim}_X p}$ can be a <u>system of parameters for X at p</u>

<u>Dimensions of fibers:</u>

<u>Main Thm:</u> Suppose $\pi: X \to Y$ is a morphism of irred. k-varieties, with $\dim X = m$, $\dim Y = n$.

Then $\dim \pi^{-1}(y) \geq m-n$ for any $y \in Y$ with $\pi^{-1}(y) \neq \emptyset$.

Moreover, there exists a nonempty open $U \subset Y$ s.t. for all $y \in U$, either $\pi^{-1}(y) = \emptyset$ or $\pi^{-1}(y)$ has pure dimension $m-n$.

(Even more info: $\dim \pi^{-1}(y)$ is an upper semicontinuous function of y.)

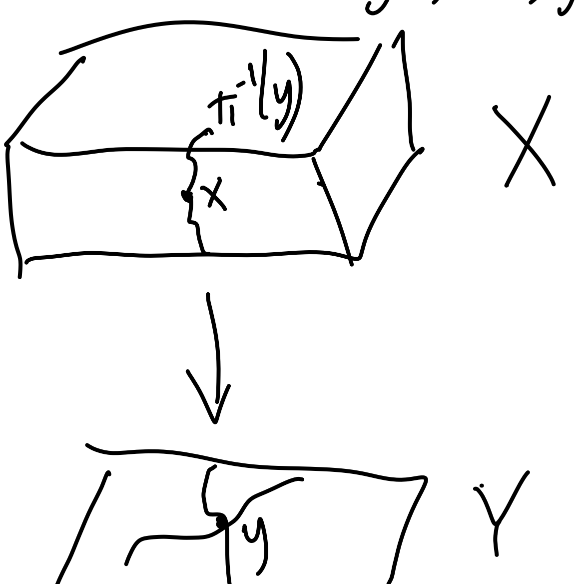
Pf of $\dim \pi^{-1}(y) \geq \dim X - \dim Y$.

For simplicity, assume y is closed. Suppose $x \in \pi^{-1}(y)$ and suppose for contradiction that

$$l = \operatorname{codim}_{\pi^{-1}(y)} x < \dim X - \dim Y = m-n.$$

Then take $f_1, \dots, f_l \in \mathcal{O}_{\underbrace{\pi^{-1}(y)}_{\text{reduced closed subscheme of } X}, x}$ a system of params

and $g_1, \dots, g_n \in \mathcal{O}_{Y,y}$ a system of params.



Want system of params in $\mathcal{O}_{X,x}$ of length $l+n<m$ for contradiction.

Have morphisms $\mathcal{O}_{Y,y} \xrightarrow{\pi^*} \mathcal{O}_{X,x}$

$$\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\pi^{-1}(y),x}$$

(loc of $A \twoheadrightarrow A/I$)

Take lifts $\tilde{f}_1, \dots, \tilde{f}_l \in \mathcal{O}_{X,x}$

and images $\pi^* g_1, \dots, \pi^* g_n \in \mathcal{O}_{X,x}$.

Can check: these cut out $x \in X$, $\Rightarrow\Leftarrow$ $\blacksquare$