Today:
- finish up with rational maps
- proper morphisms briefly
- start on dimension (last major topic this term)
- notes on schedule for rest of term.

Recall: a rat map $X \dashrightarrow Y$

was a morphism
$$\begin{tikzcd} X \arrow[r, dashed] & Y \\ U \arrow[u, hook, "\text{dense open}"] \arrow[ur] & \end{tikzcd}$$

Last time: if X is reduced and Y is separated, then rat. maps have a well-defined domain of definition.

Today: Vakil 6.5.6.

$\mathbb{C}(t, u, \sqrt{t+u^3})$

Thm: Let k be a field. Then there is an (arrow-reversing) equivalence of categories

adjoin finitely many variables + relations

$$\begin{pmatrix} \text{irred. } k\text{-varieties} \\ \underbrace{\text{dominant}}_{\text{dense image}} \text{ rational maps} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{f.g. field extensions of } k, \\ k\text{-alg homomorphisms} \end{pmatrix}$$

$$X \longmapsto \underline{K(X)} = \mathcal{O}_{X,\eta}$$

$$f: X \dashrightarrow Y \longmapsto \text{induced map } \mathcal{O}_{Y,\eta'} \hookrightarrow \mathcal{O}_{X,\eta}$$

(Idea: open nbhds of η = dense opens (in irred. scheme))

(Need to check: given f.g. field extension K/k, take f.g. k-algebra A with $K(A) = A_{(0)} = K$, and then Spec A works "on left", $K(\operatorname{Spec} A) = K$.
then also need to lift morphisms)

Cor: Two irred. k-varieties X, Y have isomorphic nonempty open subschemes $\Longleftrightarrow \underbrace{K(X)}_{\mathcal{O}_{X,\eta}} \cong K(Y)$.

$X = \mathbb{A}^1_k$

$Y = \operatorname{Spec} k(t)$ is not a counterexample because Y is not finite type over k.

Def: A morphism $\pi: X \to Y$ is universally closed if every base change $X \times_Y Z \to Z$ is closed, i.e sends closed sets to closed sets.

Def: A morphism is proper if it is separated, finite type, and universally closed.

Intuition: "classical" top. analogue is a cont. map $f: X \to Y$ s.t. if $Z \subseteq Y$ is compact, then so is $f^{-1}(Z)$. ("compact fibers").

$X \to \operatorname{Spec} \mathbb{C}$ is proper $\Longleftrightarrow$ "classical points of X are compact Hausdorff"

Examples:

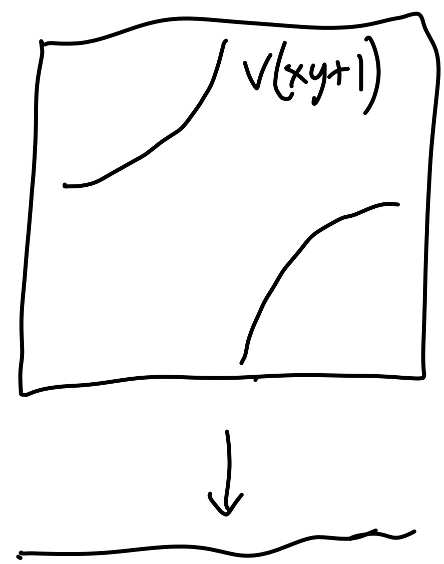
1) Finite morphisms are proper
(finite $\Rightarrow$ finite type, finite $\Rightarrow$ affine $\Rightarrow$ separated,
recall finite morphisms are closed, hence univ. closed)

In particular, closed embeddings are proper.

2) $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not proper, because

$p_2 : \mathbb{A}^2_k \to \mathbb{A}^1_k$ is not closed,

since $p_2(V(xy+1)) = D(t) = \mathbb{A}^1_k - \{0\}$.



(intuition: $\mathbb{C}$ is not compact.)

3) $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper. ($\mathbb{C}P^1 = S^2$ is compact)

Enough to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is closed.

This is a somewhat tricky alg. result (Thm 7.4.7 in Vakil).

4) Compositions of proper morphisms are proper, so proj. A-schemes are proper over Spec A.

<u>Remark</u>: Not all proper k-schemes are proj, but it is messy to construct a counterexample.

<u>Example of using properness</u>:

Prop: Let $k = \bar{k}$ and let X be a connected, proper $\underbrace{k\text{-variety}}_{\text{reduced } k\text{-scheme}}$. Then $\mathcal{O}_X(X) = k$.

Pf: $f \in \mathcal{O}_X(X) \rightsquigarrow \pi : X \to \mathbb{A}^1_k$.

Let π' be the composition $X \to \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$.

Then
$$\begin{tikzcd} X \arrow[r, "\pi'"] \arrow[dr, "\text{proper}"'] & \mathbb{P}^1_k \arrow[d, "\text{separated}"] \\ & \operatorname{Spec} k \end{tikzcd}$$
, so by factorization,

$\pi' : X \to \mathbb{P}^1_k$ is proper, hence the set-theoretic image of π' is closed.

But $\operatorname{im}(\pi') \subseteq \mathbb{A}^1_k \subset \mathbb{P}^1_k$ and is connected (because X is conn.), hence is a single closed point $p \in \mathbb{A}^1_k$.

So the scheme-theoretic image is also just p as a set and is reduced since X is reduced, so is just $\operatorname{Spec} k = p \hookrightarrow \mathbb{P}^1_k$. So $\pi' : X \to \mathbb{P}^1_k$ factors through $\operatorname{Spec} k$, so is just $X \xrightarrow[\text{structure morphism}]{} \operatorname{Spec} k \hookrightarrow \mathbb{P}^1_k$. $\blacksquare$

If $k \neq \bar{k}$, just got

$$\begin{tikzcd}
X \arrow[rr] \arrow[drr] \arrow[ddr] & & \mathbb{A}^1_k \\
& & \operatorname{Spec} k_p \arrow[u] \arrow[dl] \\
& \operatorname{Spec} k &
\end{tikzcd}$$

Q: What are the morphisms $X \to \operatorname{Spec} k_p$?

Unclear if $k_p \neq k$.

Hard fact: finite $\Longleftrightarrow$ affine + proper.

Dimension:

Expectations:

0) Any nonempty scheme X has a well-defined "dimension" $\dim X \in \{0, 1, \ldots, +\infty\}$

1) $\mathbb{A}^n_k$, $\mathbb{P}^n_k$ have dimension n.

2) $V(f) \subseteq X$ "usually" has dimension $\dim X - 1$.

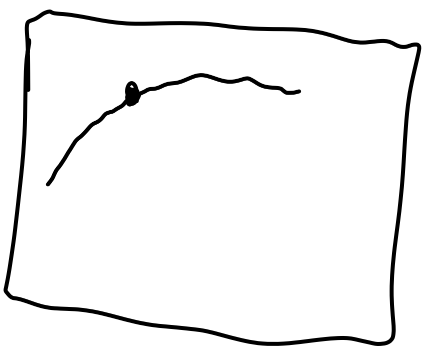
3) If X is a $\mathbb{C}$-variety, $\dim X$ should be equal to the dimension of $X(\mathbb{C})$ "as an analytic space/$\mathbb{C}$" (complex dimension, not real dimension)

Miracle: $\dim X$ only depends on the underlying top. space of X. (e.g. $\dim X = \dim X^{red}$)

<u>Def</u>: The <u>dimension</u> of a top. space X, denoted $\dim X$, is the supremum of all $d \geq 0$ s.t. there exists a chain

$$Z_0 \subset Z_1 \subset \cdots \subset Z_d \subseteq X$$

of distinct closed irred. $Z_i \subseteq X$.



"point $\subset$ curve $\subset$ surface" $\rightsquigarrow$ dim 2.

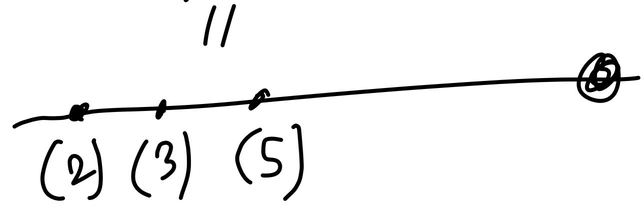
<u>Def</u>: If A is a ring, let $\dim A = \dim \operatorname{Spec} A$
$=$ sup. of lengths of chains of prime ideals in A.

This is also called the <u>Krull dimension</u> of A.

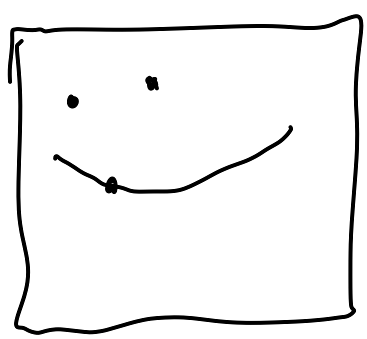
Examples:

0) Spec k has dim 0. More generally, if X is a finite k-scheme, then $\dim X = 0$ (since X is discrete).

1) $\mathbb{A}^1_k$ has dim 1, as does Spec $\mathbb{Z}$.



2) $\mathbb{A}^2_k$ and Spec $\mathbb{Z}[t]$ have dim 2.



3) Not easy to describe all closed irreds in $\mathbb{A}^d_k$ for $d \geq 3$, so no direct way to compute dimension.

4) Finite morphisms preserve dimension.

<u>Thm:</u> If $f: X \to Y$ is finite, $\dim X = \dim \pi(X)$.

<u>Pf:</u> comm. alg.

Recall: <u>Lying Over Thm:</u> If $\varphi: B \hookrightarrow A$ is an integral extension and $q \subset B$ is prime, then there exists $\overset{\text{prime}}{p} \subset A$ with $\varphi^{-1}(p) = q$ (i.e. $\varphi^*: \operatorname{Spec} A \to \operatorname{Spec} B$ is surjective).

A consequence:

<u>Going Up Thm:</u> If $\varphi: B \to A$ is an integral homomorphism, $q_0 \subset q_1 \subset \cdots \subset q_d \subset B$ are prime, and $\overset{\text{prime}}{p_0} \subset A$ satisfies $\varphi^{-1}(p_0) = q_0$, then there exists a chain of prime ideals $p_0 \subset p_1 \subset \cdots \subset p_d \subset A$ with

$$\varphi^{-1}(p_i) = q_i \text{ for all } i.$$

Schedule:

Nov 19: dimension (Ch. 11/12)

next week: break, no scheduled office hours but feel welcome to e-mail me

Dec 1: dimension (Ch. 11/12)

Dec 3:
Dec 8:] Sketch approaches/proofs to a sequence of thms about curves and projective varieties (e.g. Thm 16.5.1)

Final problem set will be posted on Nov 19 and due Dec 8.

It will be a little longer/harder, and deal with various dimension-related topics so might want to wait until Dec 1 for some of it.