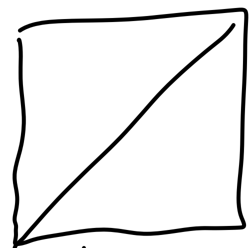
Last time: factorization

$$\begin{tikzcd}
 & & Z & & \\
 & X \arrow[ur] \arrow[rr, "\pi", "p"'] \arrow[dr, hook, "\Gamma_\pi", "\text{closed emb. } p"'] \arrow[ddl] & & Y \arrow[ul] \arrow[dr] & \\
 & & X \times_Z Y \arrow[ur, "p"'] \arrow[dr] \arrow[ddl] \arrow[rr, phantom, "\square"] & & Z \\
Y \arrow[dr, hook, "\Delta", "\text{closed emb.}"'] & & & X \arrow[ur, "p"'] & \\
 & Y \times_Z Y & & &
\end{tikzcd}$$



"x−y=0"

Def: A morphism $Y \to Z$ is separated if $\Delta : Y \to Y \times_Z Y$ is a closed embedding.

(Ex: $\mathbb{A}^1_k \to \operatorname{Spec} k$ is separated, $\mathbb{A}^1_k$ is a separated k-scheme)

<u>Factorization Lemma:</u> Suppose that P is a property of morphisms that is closed under composition and base change and satisfied by closed embeddings. Suppose that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right] & Y \arrow[r, "g"] & Z \end{tikzcd}$$ satisfy:

i) g is separated

ii) $g \circ f$ has P.

Then f has P.

"morphisms to separated schemes behave well"

Pf: previous factorization diagram. $\blacksquare$

Example: a morphism of A-schemes from an affine A-scheme to a separated A-scheme is affine.

When is
$$\left(\mathrm{Spec}\, B \longrightarrow X \right)$$
an affine morphism?

Answer: "almost always"

Which things are separated?

<u>Lemma</u>: Any morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is sep.

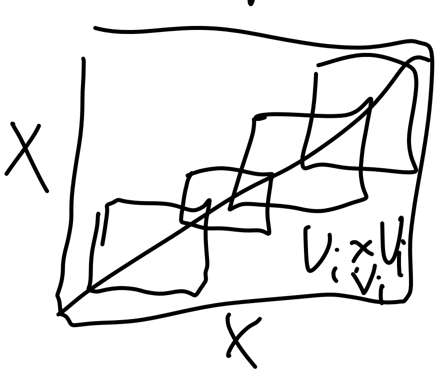
<u>Pf</u>: $\Delta: \operatorname{Spec} A \longrightarrow \operatorname{Spec} A \otimes_B A$ is a closed embedding because
$A \otimes_B A \longrightarrow A$ is surjective. $\blacksquare$

$a_1 \otimes a_2 \longmapsto a_1 a_2$

<u>Cor</u>: A morphism $\pi: X \to Y$ is separated
$\iff \Delta: X \to X \times_Y X$ has closed image.

<u>Pf</u>: The image of Δ can be covered by $U \times_V U$ with U, V affine opens, so Δ is a closed embedding into some open subscheme of $X \times_Y X$. $\blacksquare$ "locally closed subscheme"

if X = top. space,
$\Delta: X \to X \times X$
has closed image
$\iff X$ is
Hausdorff



$X \times_Y X$

$$X \times_Y X = \bigcup_{V_i \text{ covering } Y} \pi^{-1}(V_i) \times_{V_i} \pi^{-1}(V_i)$$

Cor: Affine morphisms are separated. ♥

More facts about separatedness:

1) open embeddings are separated
2) separated morphisms are closed under composition and base change.
3) morphisms between separated $\mathbb{Z}$-schemes are separated (factor. lemma). ♥

Prop: $\mathbb{P}^n_A \longrightarrow \operatorname{Spec} A$ is separated.

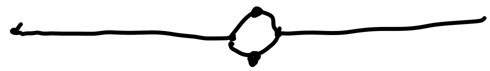
((Cor: (quasi)projective A-schemes are separated))

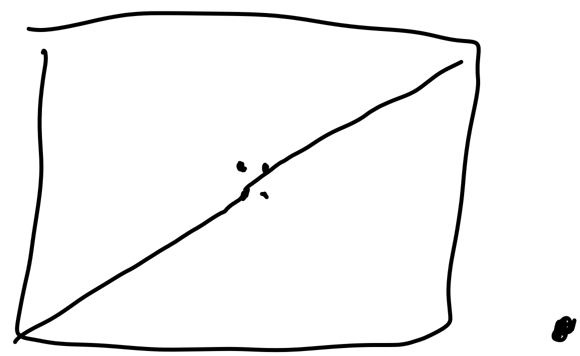
Pf:

$$\begin{tikzcd}
 & \mathbb{P}^n \ [x_0 : \cdots : x_n] \arrow[dl, "V_2"'] \arrow[dr, "\Delta"] \arrow[dd] & \\
\mathbb{P}^{\binom{n+2}{2}-1} \ [x_0^2 : x_0x_1 : \cdots : x_n^2] \arrow[dr, "L"'] & & \mathbb{P}^n \times \mathbb{P}^n \ ([x_0 : \cdots : x_n], [x_0 : \cdots : x_n]) \arrow[dl, "s"] \\
 & \mathbb{P}^{n^2+2n} \ [x_0^2 : x_0x_1 : \cdots : x_1x_0 : \cdots : x_n^2] &
\end{tikzcd}$$

will commute for
V_2 = Veronese
s = Segre
L = linear map.

V_2, L, s are all closed embeddings, so Δ is by factorization

Example of a nonseparated morphism:
line with two origins  $\longrightarrow \operatorname{Spec} k$.



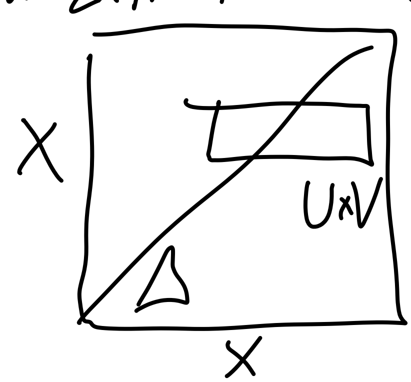
Def: A k-scheme X is a <u>variety</u> (or <u>k-variety</u>) if it is <u>reduced</u>, <u>separated</u>, and <u>finite type</u>.

$\underbrace{\text{reduced}}_{\text{"no fuzz"}}$, $\underbrace{\text{separated}}_{\text{"Hausdorff"}}$, $\underbrace{\text{finite type}}_{\text{"finite-dimensional"}}$.

Recall:
($\operatorname{Spec} k(t) \longrightarrow \operatorname{Spec} k$ is not finite type)

Lemma: Morphisms of k-varieties (i.e. of k-schemes) are separated and finite type.

Pf: factorization lemma.

(Aside: 

"$U \times V \cap \Delta = U \cap V$"
$\rightsquigarrow$ sep $\Longrightarrow$ quasi. sep.)

<u>Def</u>: A <u>rational map</u> $\pi: X \dashrightarrow Y$ is an equiv. class of pairs $(U, f: U \to Y)$, where U is a dense open in X and the equivalence relation is gen. by $(U, f) \sim (V, f|_V)$ for $V \subseteq U$

("two rational maps are equal if they agree on some dense open set")

<u>Example</u>: rational maps of k-schemes

$$\mathbb{A}^1_k \dashrightarrow \mathbb{A}^1_k \text{ correspond to}$$

rational functions $f \in k(t)$

$\left(\frac{t}{(t-2)(t-3)}\right.$ is a morphism $\left.\mathbb{A}^1_k - \{2,3\} \to \mathbb{A}^1_k\right)$.

More generally, if X is an integral k-scheme, then the function field $K(X)$ is also the set of rational maps (of k-schemes)

$$X \dashrightarrow \mathbb{A}^1_k.$$

True for $X = \mathbb{P}^1_k$, false in general, will be true for X a "smooth curve."

<u>Q</u>: When is it the case that every rational map $X \dashrightarrow \mathbb{A}^1_k$ extends to a morphism $X \longrightarrow \mathbb{P}^1_k$?

False: $X = \mathbb{A}^2$, $\pi = \frac{x}{y}$.

"functions from reduced schemes behave well"

<u>Reduced-to-Separated Thm</u>: Suppose X is a reduced Z-scheme, Y is a separated Z-scheme, and $f, g : X \to Y$ are morphisms of Z-schemes that agree on some dense open U in X. Then $f = g$.

"uniqueness of extension from U to X".

<u>Pf</u>: Let W be the fiber product

$$\begin{tikzcd} W \arrow[r] \arrow[d, hook] & Y \arrow[d, hook, "\Delta"] \\ X \arrow[r, "(f,g)"'] & Y \times_Z Y \end{tikzcd}$$

, so $W \to X$ is a closed embedding.

$\boxed{\text{We think of } W \hookrightarrow X \text{ as the "largest closed subscheme where } f \text{ and } g \text{ agree"}}$.

$\exists !$

<u>Claim</u>:

$$\begin{tikzcd} U \arrow[r, "f|_U = g|_U"] \arrow[d, hook] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "(f,g)"'] & Y \times_Z Y \end{tikzcd}$$

Y sep.

commutes. Given the claim, $\boxed{U \hookrightarrow X \text{ factors through } W \hookrightarrow X.}$

So $\text{image}(W \hookrightarrow X) \supseteq \text{image}(U \hookrightarrow X)$

$\text{image}(W \hookrightarrow X)$ = closed subset, $\text{image}(U \hookrightarrow X) = U$ = dense open,

So as a closed subset, $W = X$. Since X is reduced, this means $W \hookrightarrow X$ is the identity.

Conclusion: fiber square

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, equal] \arrow[drr, bend left, "h"] & Y \arrow[d, "\Delta"] & \\
X \arrow[r, "{(f,g)}"] \arrow[rr, bend right, "f"'] & Y \times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"] & Y \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}$$

For this to commute, must have $f = h = g$. $\blacksquare$

Cor/Def: Any rational map from reduced to separated has a well-defined maximum domain of definition

Pf: Take union of $U \subseteq X$ for all $(U, f: U \to Y)$ in the equiv. class of the rat. map.

By prev. theorem, all the f's glue together. $\boxtimes$

$f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$

$f: X \dashrightarrow Y \rightsquigarrow f: \underbrace{U}_{\text{domain of def.}} \to Y$