Last time: fiber products

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

On affines: $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B = \operatorname{Spec}(A \otimes_C B)$,

glues well to give construction in general.

Note: many properties of morphisms that can be expressed in terms of affine opens are easily seen to be preserved under base change. (since preserved by $\otimes$)

Some examples:

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{C} \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{R} \end{tikzcd}$$

"disconnected fibers"

"connected fibers"

Today: 3 separate topics about $X \times_Z Y$.

I) Scheme-theoretic fibers:

$$\begin{tikzcd} X \times_Y \{y\} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \{y\} \arrow[r] & Y \end{tikzcd}$$

$X \times_Y \{y\} = \pi^{-1}(y)$

π function $\rightsquigarrow$

$$\begin{tikzcd} X \times_Y \operatorname{Spec} k_y \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \operatorname{Spec} k_y \arrow[r] & Y \end{tikzcd}$$

<u>Def</u>: $X \times_Y \operatorname{Spec} k_y$ is the <u>scheme-theoretic fiber</u> of π over $y \in Y$.

<u>Prop</u>: $X \times_Y \operatorname{Spec} k_y \longrightarrow X$ is a homeom onto its image, which is $\pi^{-1}(y)$.

(This justifies notation "$\pi^{-1}(y) = X \times_Y \operatorname{Spec} k_y$")

<u>Pf</u>: Restrict to affine open $U \subseteq X$ and check algebra. ∎

Cor:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd} \qquad x \in X, \quad y \in Y, \quad x \mapsto z, \quad y \mapsto z$$

$$\left\{ w \in X \times_Z Y \;\middle|\; \begin{matrix} p_1(w) = x, \\ p_2(w) = y \end{matrix} \right\}$$

$\cong$ (bijective)

$$\operatorname{Spec} \left(k_x \otimes_{k_z} k_y \right)$$

Galois theory.

Pf:

$$\begin{tikzcd} p_1^{-1}(x) \cap p_2^{-1}(y) \arrow[r] \arrow[d] & p_1^{-1}(x) \arrow[r] \arrow[d] & \operatorname{Spec} k_x \arrow[d] \\ p_2^{-1}(y) \arrow[r] \arrow[d] & X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k_y \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

$p_1^{-1}(x) \cap p_2^{-1}(y) \cong \operatorname{Spec} k_x \times_Z \operatorname{Spec} k_y$. $\blacksquare$

Cor: if $k_x = k_y = k_z$, get exactly one element in this set (since $\operatorname{Spec} k \otimes_k k = \operatorname{Spec} k = \{*\}$).

Example: $\pi : \mathbb{A}^1_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$

$t \mapsto t^2$

Get a k_p-scheme $\pi^{-1}(p)$ for each $p \in \mathbb{A}^1_{\mathbb{R}}$.

Five different isom classes of fiber.

Defs: A geometric point of a k-scheme X is a morphism of k-schemes $\operatorname{Spec} K \to X$, where K is an alg. closed field.

A geometric fiber of a morphism $\pi : X \to Y$ (of k-schemes) is the base change by a geometric point of Y

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\text{geom fiber}"'] & X \arrow[d, "\pi"] \\ \operatorname{Spec} K \arrow[r] & Y \end{tikzcd}$$

A geom. point is the data of a point $y \in Y$ along with some inclusion $k_y \hookrightarrow K = \overline{K}$.

$$\begin{tikzcd} W \arrow[r, "\tilde{\iota}"] \arrow[d] & W \arrow[r] \arrow[d] & \pi^{-1}(y) \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \operatorname{Spec} K \arrow[r, "\iota"] & \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k_y \arrow[r] & Y \end{tikzcd}$$

Some properties of schemes (connected, irreducible, reduced, integral)

are properly thought of as props of geom. fibers.

(Ex: Def: $\pi: X \to Y$ is <u>geom. connected</u> if every geometric fiber of X is connected.

So $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ is not geom. connected,

but $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}$ is geom. connected,

i.e. $\operatorname{Spec} \mathbb{C}$ is geom. connected as a $\mathbb{C}$-scheme, but not as an $\mathbb{R}$-scheme.

Geom. fiber perspective: as an $\mathbb{R}$-scheme, $\operatorname{Spec} \mathbb{C}$ is 2 points squished together but still geom. distinct (and thus disconnected))

Fact: being geom. $\left\{\begin{matrix}\text{connected} \\ \text{irred.} \\ \text{reduced}\end{matrix}\right\}$ is preserved by base change.

II) Products of proj. schemes.

Suppose $\mathrm{Proj}\, S_\bullet$ and $\mathrm{Proj}\, T_\bullet$ are proj. A-schemes

(S, T are f.g. graded rings over $A = S_0 = T_0$).

<u>Question</u>: Is $\mathrm{Proj}\, S_\bullet \times_{\mathrm{Spec}\, A} \mathrm{Proj}\, T_\bullet$ also a proj. A-scheme?

<u>Prop</u>: $\mathrm{Proj}\, S_\bullet \times_{\mathrm{Spec}\, A} \mathrm{Proj}\, T_\bullet \cong \mathrm{Proj}\left(\overbrace{\underbrace{\bigoplus_{n \geq 0} (S_n \otimes_A T_n)}_{\text{can choose gens}}}^{\text{graded ring}}\right)$.

Special case of $\mathbb{P}^m \times \mathbb{P}^n$: we want a closed embedding $\mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^N$

Classically: $([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \longmapsto [f_0 : f_1 : \cdots : f_N]$

for some polynomials $f_0, \ldots, f_N \in k[x_0, \ldots, x_m, y_0, \ldots, y_n]$.

In order for this to be well-defined, need f_i to be bihomogeneous (simult. homog. in x's and y's).

i.e. $f_i(\lambda_1 x_0, \ldots, \lambda_1 x_m, \lambda_2 y_0, \ldots, \lambda_2 y_n)$

$$= \lambda_1^{d_1} \lambda_2^{d_2} f_i(x_0, \ldots, x_m, y_0, \ldots, y_n).$$

Simplest way: $d_1 = d_2 = 1$:

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \longmapsto [x_0 y_0 : x_0 y_1 : \cdots : x_m y_n]$$

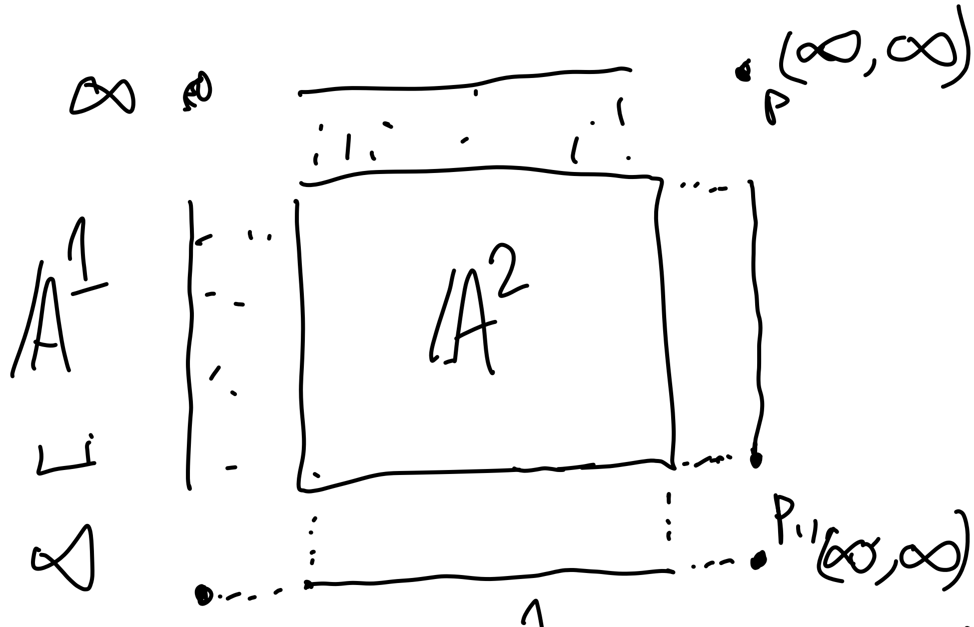
This defines a map $\mathbb{P}^m \times \mathbb{P}^n \longrightarrow \underbrace{\mathbb{P}^{mn+m+n}}$,
called the <u>Segre embedding</u>.

Fact: This is a closed embedding of schemes

$$S: \mathbb{P}^m_A \times_{\operatorname{Spec} A} \mathbb{P}^n_A \hookrightarrow \mathbb{P}^{mn+m+n}_A.$$

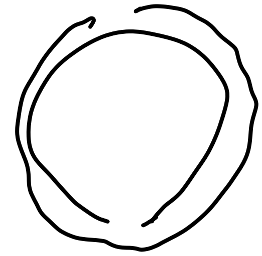
<u>Cor</u>: closed subschemes of $\mathbb{P}^m \times \mathbb{P}^n$ are cut out by "bihomogeneous ideals" in $k[x_0, \ldots, x_m, y_0, \ldots, y_n]$.

$\mathbb{P}^1_k \times_{\mathrm{Spec}\,k} \mathbb{P}^1_k$: expect covered by 4 affine opens, each isom to $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$.



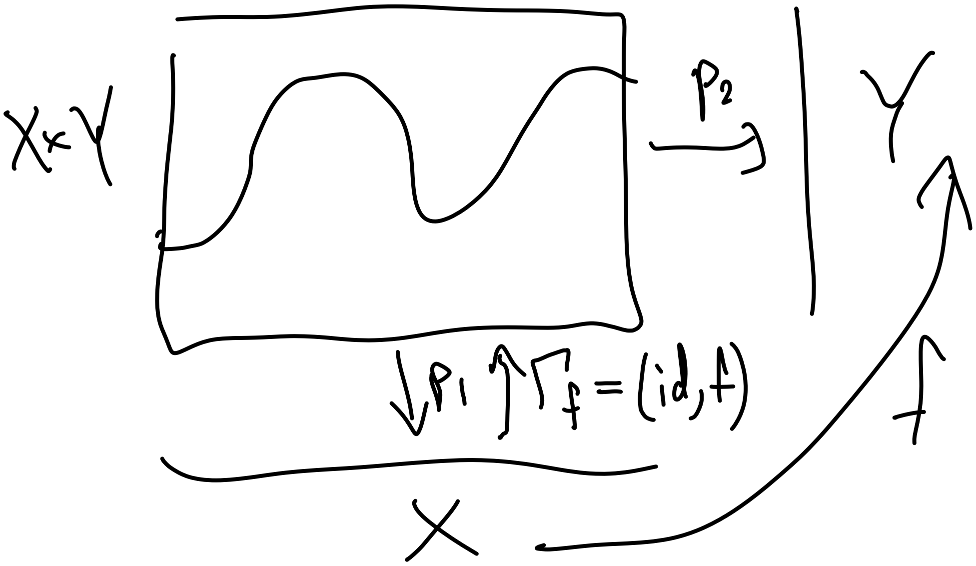
$\mathbb{A}^1 \sqcup \infty = \mathbb{P}^1$

$\mathbb{P}^1_{\mathbb{R}}$



III) Graph morphisms:

Idea: if $f: X \to Y$ is a function of sets, it factors through its graph in $X \times Y$:



$f = p_2 \circ \Gamma_f$.

Def: Let $\pi: X \to Y$ be a morphism of Z-schemes

a Z-scheme is a morphism $X \to Z$, and a morphism of Z-schemes is a comm triangle

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd},$$

so "(Spec A)-scheme" = "A-scheme"

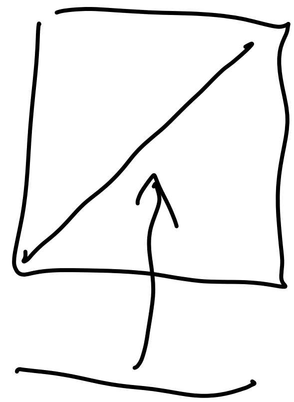
The graph morphism of π is $\Gamma_\pi = (id, \pi): X \to X \times_Z Y$ defined by the univ. prop. of $X \times_Z Y$:

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "\pi"'] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

Notes: 1) In particular, $\pi : X \to Y$ factors as

$$\begin{tikzcd} X \arrow[r, "\Gamma_\pi"] \arrow[rr, bend right, "\pi"'] & X \times_Z Y \arrow[r, "p_2"] & Y \end{tikzcd}$$

2) Important special case: $\pi = id : X \to X$, then $\Gamma_\pi = \Delta_{(X \to Z)} : X \to X \times_Z X$.



One nice feature of the graph morphism factorization $\pi = p_2 \circ \Gamma_\pi$: both p_2 and Γ_π are base changes of "simpler" morphisms

$$
\begin{tikzcd}
X \arrow[rr, bend left, "\pi"] \arrow[dr, "\Gamma_\pi"] \arrow[dd] & & Y \arrow[dr] & \\
& X \times_Z Y \arrow[ur, "p_2"] \arrow[dr] \arrow[dd] \arrow[r, phantom, "\diamond"] \arrow[l, phantom, "\square"] & & Z \\
Y \arrow[dr, "\Delta_{(Y \to Z)}"'] & & X \arrow[ur] & \\
& Y \times_Z Y & &
\end{tikzcd}
$$

$\boxed{Y \to Z}$

$\rightsquigarrow$

Consequence: if $X \to Z$ and $\boxed{\Delta : Y \to Y \times_Z Y}$ have nice properties, then so does every morphism of Z-schemes $X \to Y$.