Fiber products of schemes:

Given
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
, want
$$\begin{tikzcd} \boxed{X \times_Z Y} \arrow[r, "\boxed{p_1}"] \arrow[d, "\boxed{p_2}"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

base change

universal for such diagrams, i.e. any
$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
should factor uniquely through via a morphism $W \to X \times_Z Y$

Another way to say this

$$Mor(W, X \times_Z Y) \cong \left\{ (\alpha, \beta) \;\middle|\; \begin{array}{l} \alpha \in Mor(W, X) \\ \beta \in Mor(W, Y) \\ f\alpha = g\beta \end{array} \right\}$$

Language:
p_1 is the base change or pullback of g by f.

Q: Does $X \times_Z Y$ always exist in Sch?

Fiber products always exist in Set or Top:

$$\left[X \times_Z Y := \{ (x,y) \in \underbrace{X \times Y} \mid f(x) = g(y) \}. \right]$$

Tempting to try to promote this construction from Top to Sch. But this is doomed to failure. Construction will be harder in Sch.

Thm: For any $X, \underbrace{Y, Z, f: X \to Z, g: Y \to Z}_{\in \mathrm{Sch}}$,

$X \times_Z Y$ exists.

(Note: underlying set of $X \times_Z Y$ taken in Sch is usually not the same as $X \times_Z Y$ taken in Sets)

(Example:
$$\begin{tikzcd}
\mathbb{A}^2_k \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\
\mathbb{A}^1_k \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$
is a "fibered square" but points in $\mathbb{A}^2_k$ are more interesting than the setwise fiber product)

Plan for today:

1) some motivation
2) sketch of pf of above thm (construction of $X \times_Z Y$)
3) start on examples

Motivation:

1) "product": [in Top:
$$\begin{tikzcd} X\times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \{*\} \end{tikzcd}$$
is a fibered square]

We would like a construction in Sch that mirrors cartesian product on the "classical" points.

Given k-schemes X_1, X_2, we might want a k-scheme X_3 with morphisms to X_1 and X_2 such that

$$\begin{tikzcd} X_3(k) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\|"] & X_1(k)\times X_2(k) \arrow[d] \arrow[dr] & \\ \mathrm{Mor}_{Sch_k}(\mathrm{Spec}\, k, X_3) & X_1(k) & X_2(k), \end{tikzcd}$$

The fiber product $X_3 = X_1 \times_{\mathrm{Spec}\, k} X_2$ using the given morphisms $X_i \to \mathrm{Spec}\, k$ has this property.

(More generally: $\left(X \overset{Sch}{\underset{Z}{\times}} Y\right)(A) = X(A) \overset{Set}{\underset{Z(A)}{\times}} Y(A)$)

(Summary; we will have things like $\mathbb{P}^1 \times \mathbb{P}^2$

$$\mathbb{P}^1 \times \mathbb{P}^2 = \mathbb{P}^1_k \underset{\operatorname{Spec} k}{\times} \mathbb{P}^2_k$$).

(Warning: $\mathbb{A}^2_k = \mathbb{A}^1_k \underset{\operatorname{Spec} k}{\times} \mathbb{A}^1_k$ is only true on the level of sets on classical points).

2) "base change": we want a canonical way of changing our "base ring", e.g.

$$\begin{tikzcd} X_1 = \operatorname{Spec} k[x_1,\dots,x_n]/f(x_1,\dots,x_n) \arrow[d, squiggly, shift left=2, "\text{"base change"}"] \\ X_2 = \operatorname{Spec} \ell[x_1,\dots,x_n]/f(x_1,\dots,x_n) \arrow[u, shift left=2] \end{tikzcd}$$

for ℓ a field extension of k.

This will be accomplished by

$$X_2 = X_1 \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} \ell$$

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & \operatorname{Spec} \ell \arrow[d] \\ X_1 \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\bigcirc"] & X \arrow[d] \\ \operatorname{Spec} l \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$\longleftrightarrow$ W is an l-scheme and $W \to X$ is a morphism of k-schemes.

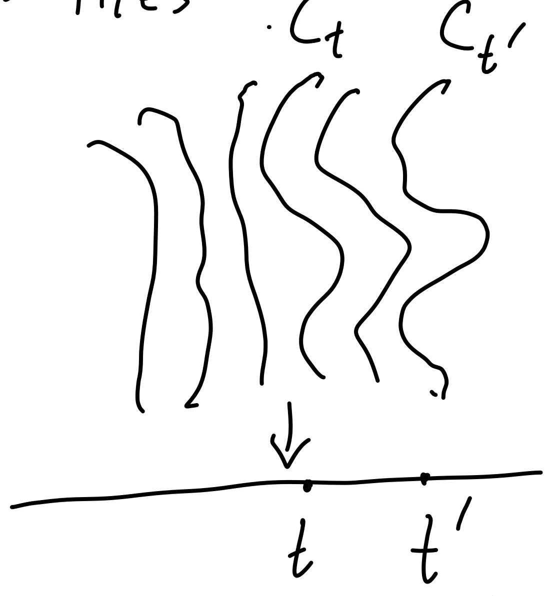
Another example: If X is a $\mathbb{Z}$-scheme, then $X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$ will be an $\mathbb{F}_p$-scheme.

$\uparrow$ finite field of order p

3) "pullback": "pulling back families":

Given morphism "whose fibers have same feature"

$$\begin{tikzcd} X \arrow[d] \\ Z \end{tikzcd}$$



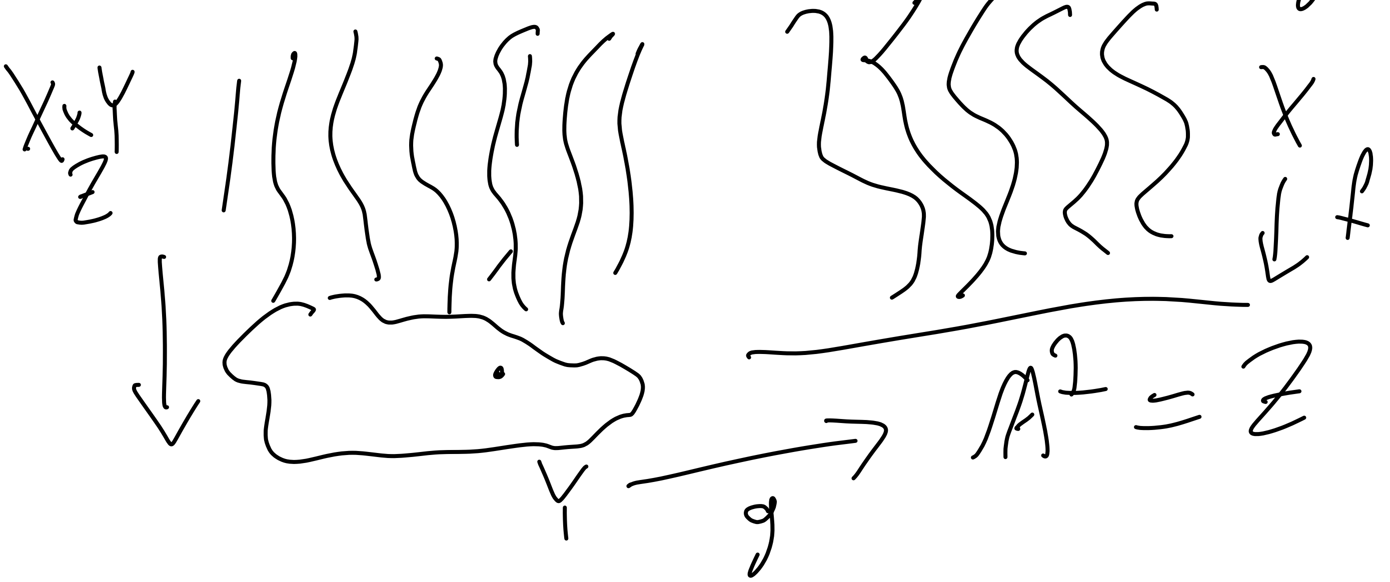
$$\begin{tikzcd} f: (x,y,t) \arrow[d, mapsto] & X = V(y^2-x^3-x-t) \subseteq \mathbb{A}^3_k \arrow[d] \\ t & Z = \mathbb{A}^1_k \end{tikzcd}$$

"family of cubic plane curves param by Z"

So (classically) fibers of f are cubic plane curves

$$C_t := f^{-1}(t) = V(y^2-x^3-x-t) \subseteq \mathbb{A}^2_k$$

Given this family over Z, along with a morphism $g: Y \to Z = \mathbb{A}^1$, we should be able to pull back the family along g to get a family of cubic plane curves param by Y: the fiber over $y \in Y$ should look like $C_{g(y)}$



$$\begin{tikzcd}
X \times Y \arrow[d, "\text{pulled back family.}"'] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

Idea of construction of $X \times_Z Y$:

1) on affines: If we assume all of W, X, Y, Z are affine (i.e fiber product in category of affine schemes),

then morphisms between affine schemes $\longleftrightarrow$ morphisms between rings (opp. direction),

so fiber product $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$ should be $\operatorname{Spec} D$ satisfying universality in Rings for

$$\begin{tikzcd}
D \arrow[dr, phantom, "\circ"] & A \arrow[l] \\
B \arrow[u] & C \arrow[l] \arrow[u]
\end{tikzcd}$$

"fibered coproduct"

This D is the tensor product $A \otimes_C B$.

(construction: add. gen by $a \otimes b$ with a bunch of relations)

Lemma: $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \cong \operatorname{Spec} A \otimes_C B$.

Pf: Recall that $\operatorname{Mor}_{Sch}(X, \operatorname{Spec} A)$
$$= \operatorname{Mor}_{Rings}(A, \mathcal{O}_X(X)).$$ ▨

2) Passing to open subschemes:

Lemma: Suppose $X \times_Z Y$ exists:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}.$$

Suppose U, V, W are open subschemes of X, Y, Z with $f(U), g(V) \subseteq W$. Then $U \times_W V$ exists and is the open subscheme
$$p_1^{-1}(U) \cap p_2^{-1}(V) \text{ of } X \times_Z Y.$$

Pf: Stack fibered squares

$$\begin{tikzcd}
p_1^{-1}(U) \cap p_2^{-1}(V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & p_1^{-1}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\
p_2^{-1}(V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
V \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

, plus $U \times_W V = U \times_Z V$. ▨

<u>Cor</u>: If $X \times_Z Y$ exists, it has an affine open cover by schemes of the form $U \times_W V$ as above with U, V, W all affine.

This lets us work explicitly on affines with $X \times_Z Y$.

Actual construction of $X \times_Z Y$: glue together all these affine.

Examples (all in affine setting)

1) $\mathbb{A}^m_k \times_{\operatorname{Spec} k} \mathbb{A}^n_k \cong \mathbb{A}^{m+n}_k \longleftrightarrow k[x_1,\dots,x_n] \otimes_k k[y_1,\dots,y_m]$

$\cong k[x_1,\dots,x_n,y_1,y_2,\dots,y_m]$

$\left(B \otimes_A A[t] \cong B[t]\right).$

2) $\operatorname{Spec} k[x_1,\dots,x_n]/(f) \times_{\operatorname{Spec} k} \operatorname{Spec} \ell \cong \operatorname{Spec} \ell[x_1,\dots,x_n]/f$

$\left(B \otimes_A (A/I) \cong B/(\varphi(I))\right)$

for $\varphi : A \to B$

3) $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C} = ?$

$$\begin{tikzcd}
X \arrow[rr, bend left] \arrow[r, dashed, "\exists!"'] \arrow[dr] & \ast^{\mathbb{C}} \ast^{\mathbb{C}} \arrow[r] \arrow[d] & \ast\mathbb{C} \arrow[d] \\
& \ast\mathbb{C} \arrow[r] & \ast\mathbb{R}
\end{tikzcd}$$

$\mathbb{C} \cong \mathbb{R}[t]/(t^2+1)$, so get

$\operatorname{Spec} \mathbb{C}[t]/(t^2+1) \cong \operatorname{Spec}(\mathbb{C}\times\mathbb{C}) = \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}.$

"connected fibers is not a property preserved by base change"