Last time: various things with closed subschemes of Proj S.

<u>Today</u>: 1) the Veronese embeddings $v_d : \mathbb{P}^n_k \hookrightarrow \mathbb{P}^N_k$

2) back to general theory of closed subschemes.

<u>Prop</u>: Suppose $S_\bullet = \bigoplus_{m \geq 0} S_m$ is a graded ring. Let $d > 0$. Define a new graded ring $S_{d\bullet} = \bigoplus_{m \geq 0} S_{dm}$. Then the map $S_{d\bullet} \to S_\bullet$ induces an isomorphism $\operatorname{Proj} S_\bullet \to \operatorname{Proj} S_{d\bullet}$.

<u>Pf</u>: For any homog. f of pos. degree, this morphism restricts to an isom

$$\begin{array}{ccc} D(f^d) & \xrightarrow{\sim} & D(f^d) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \operatorname{Spec}(-) & = & \operatorname{Spec}(-). \end{array}$$

$\blacksquare$

Apply this to $S_\cdot = k[x_0, \dots, x_n]$ to get that

$$\mathbb{P}^n_k = \mathrm{Proj}\, S_\cdot \cong \mathrm{Proj}\, S_{d\cdot}$$

But $S_{d\cdot}$ is gen by monomials of deg d, so we can write

$$\mathbb{P}^n_k \cong \mathrm{Proj}\ k[x_0^d, x_0^{d-1}x_1, \dots, x_n^d] / \left(\begin{matrix}\text{complicated ideal}\\ \text{of relations}\end{matrix}\right).$$

In other words, we have identified $\mathbb{P}^n_k$ with a closed subscheme of $\mathbb{P}^N_k$ for

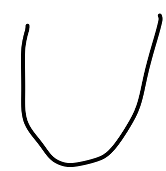
$$N = \underbrace{\binom{n+d}{d}}_{\substack{\#\text{ of monomials of deg } d\\ \text{in } n+1 \text{ variables}}} - 1$$

Def: This closed embedding

$$v_d : \mathbb{P}^n_k \hookrightarrow \mathbb{P}^N_k$$

$$[x_0 : \cdots : x_n] \longmapsto [x_0^d : x_0^{d-1}x_1 : \cdots : x_n^d]$$

is the <u>dth Veronese embedding of $\mathbb{P}^n$</u>.



Examples: 1) $v_2 : \mathbb{P}^1_k \hookrightarrow \mathbb{P}^2_k$

$$[x_0 : x_1] \longmapsto [x_0^2 : x_0x_1 : x_1^2]$$

identifies $\mathbb{P}^1_k$ with the conic $V(y_0y_2 - y_1^2)$ in $\mathbb{P}^2$.

2) $v_d : \mathbb{P}^n_k \hookrightarrow \mathbb{P}^N_k$ is more complicated.

Note that ~~conics~~ hypersurfaces $V(f) \subset \mathbb{P}^n_k$ of deg d

~~$\leftarrow \deg f = 2$~~

correspond to hyperplanes $V(\ell) \subset \mathbb{P}^N_k$

$\leftarrow \deg \ell = 1$

in the sense $v_d^{-1}(V(\ell)) = V(f)$.

Intuition: the Veronese embeddings are in some sense the "most complicated" or "universal" morphisms $\mathbb{P}^n \to \mathbb{P}^m$.

Recall: closed subscheme $Z \hookrightarrow Y$ $\;\longleftrightarrow\;$ (bijection) ideal sheaf $\mathcal{I}_Z$ on Y

$$\left(\begin{array}{c} Z \hookrightarrow Y \\ \downarrow \\ \text{"functions on } Y \text{ that vanish on pullback to } Z\text{"} \end{array} \right)$$

$\underbrace{\text{ideal sheaf } \mathcal{I}_Z \text{ on } Y}_{\text{sheaf of ideals}}$

$\mathcal{I}_Z(U) \subseteq \mathcal{O}_Y(U)$

$\underbrace{\text{"quasicoherent"}}$

$\mathcal{I}_Z(D_{\operatorname{Spec} A}(f))$
$= \mathcal{I}_Z(\operatorname{Spec} A)\left[\frac{1}{f}\right]$

Given any morphism $\pi: X \to Y$, we can define an ideal sheaf $\mathcal{I}_\pi := \ker(\mathcal{O}_Y \to \pi_* \mathcal{O}_X)$

Lemma: Given any ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$, there is a unique maximal quasicoherent ideal sheaf $\mathcal{I}' \subseteq \mathcal{I}$.

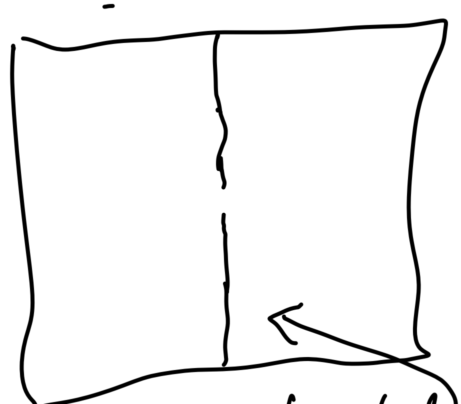
Pf: Take the sum of all of the quasicoherent ideal sheaves contained in $\mathcal{I}$. $\square$

Def: The *scheme-theoretic image* of a morphism $\pi: X \to Y$ is the closed subscheme corresponding to the quasicoherent ideal sheaf $\mathcal{I}_\pi$.

Note: This is the same thing as the "smallest closed subscheme $Z \hookrightarrow Y$ s.t. $\pi: X \to Y$ factors through Z".

Examples: 1) $\pi: \mathbb{A}^2_k \to \mathbb{A}^2_k$

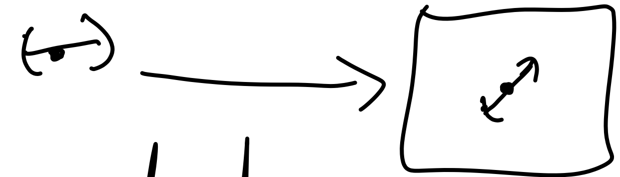
$$(x,y) \mapsto (x, xy)$$



set-theoretic image is complement of

The scheme-theoretic image of π is all of $\mathbb{A}^2_k$.

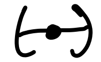
2) A morphism $\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2) \longrightarrow \mathbb{A}^2_k$

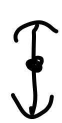


can have scheme-theoretic image either a reduced or a non-reduced point

$$k[x,y] \longrightarrow k[\varepsilon]/\varepsilon^2$$
$$x \longmapsto a+b\varepsilon$$
$$y \longmapsto c+d\varepsilon$$

$\rightsquigarrow$ scheme-theoretic image located at (a,c), reduced if $b=d=0$, otherwise non-reduced "in that direction"

$V(x^2,y) =$ 

$V(x,y^2) =$ 

$V((x-y)^2, x+y) =$ 

Lemma: If $\varphi : B \to A$ is a ring homomorphism, the image of $\varphi^* : \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ is $\mathrm{Spec}\, B/(\ker \varphi) \hookrightarrow \mathrm{Spec}\, B$.

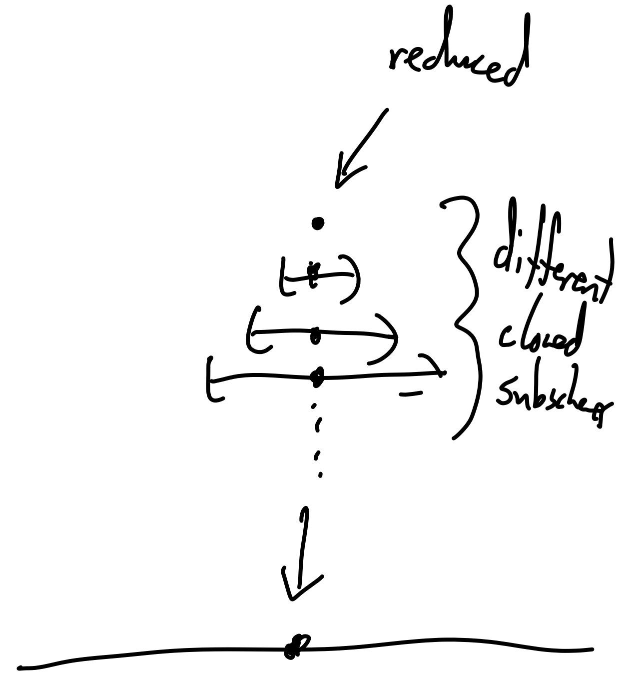
Some nice properties of scheme-theoretic image:

1) The image of a reduced scheme is reduced. (problem set)

2) In nice situations (e.g. $\pi : X \to Y$ is quasicompact or X is reduced), the underlying set of the scheme-theoretic image is equal to the top. closure of the set-theoretic image. (Thm 8.3.4 in Vakil)

What can go wrong?

$$\begin{array}{ccc} \coprod_{n>0} \operatorname{Spec} k[\varepsilon_n]/(\varepsilon_n^n) & & \varepsilon_n \\ \big\downarrow & & \big\uparrow \\ \mathbb{A}^1_k = \operatorname{Spec} k[x] & & x \end{array}$$



set-theoretic image is one closed point, but scheme-theoretic image is the entire line.

Idea: expect that given a closed subset $Z^{set} \subseteq Y$, there should exist a ~~(unique)~~ reduced closed subscheme with underlying set Z^{set}.

This will be the (induced) reduced subscheme structure on Z^{set}.

One construction: scheme-theoretic image of a morphism $X \to Y$ with image Z^{set} and with X reduced.

We can do this:

$$\coprod_{p \in Z^{set}} \operatorname{Spec} k_p \longrightarrow X$$

$$\rightsquigarrow \text{ ideal sheaf } \mathcal{I}(U) = \left\{ f \in \mathcal{O}_X(U) \,\middle|\, \begin{array}{l} f \text{ vanishes} \\ \text{at every point} \\ \text{of } Z^{set} \end{array} \right\}.$$

Another description:

$\mathcal{I}(\operatorname{Spec} A)$ is the radical ideal corresponding to the closed subset $Z \cap \operatorname{Spec} A \subseteq \operatorname{Spec} A$.

"Largest ideal cutting out $Z \cap \operatorname{Spec} A$"

__Def__: If X is a scheme, the induced reduced subscheme structure on $X \subseteq X$ defines a reduced closed subscheme $\underbrace{X^{red} \hookrightarrow X}_{\text{identity on sets}}$,

called the reduction of X.

(replace each affine open $\operatorname{Spec} A \subseteq X$ with $\operatorname{Spec}(A/\operatorname{Nil}(A))$.)