Last time: graded ring $S_\bullet$ $\rightsquigarrow$ scheme $\operatorname{Proj} S$

covered by $D(f) \cong \operatorname{Spec}\left(\left(S_\bullet\left[\frac{1}{f}\right]\right)_0\right)$.

$f \in S_d$ for $d>0$

basic examples: $\operatorname{Proj} k[\underbrace{x_0, \dots, x_n}_{\deg 1}] =: \mathbb{P}^n_k$

$\operatorname{Proj} k[x_0, \dots, x_n]/I$ = closed subscheme of $\mathbb{P}^n_k$ "cut out by I"

homog. ideal

still need to justify by constructing a morphism

$\operatorname{Proj} S_\bullet/I \hookrightarrow \operatorname{Proj} S$.

Def: A morphism of graded rings $\varphi: T_\bullet \to S_\bullet$ is a ring homomorphism such that for some integer $d \geq 0$, $\varphi(T_n) \subseteq S_{dn}$ for all $n \geq 0$.

(Example: $k[y_0, y_1] \to k[x_0, x_1]$
$$\left.\begin{array}{l} y_0 \mapsto f_0(x_0, x_1) \\ y_1 \mapsto f_1(x_0, x_1) \end{array}\right\} \text{homog of deg } d.$$
).

Lemma: A morphism of graded rings $\varphi: T_\bullet \to S_\bullet$ induces a morphism of schemes
$$\varphi^*: \mathrm{Proj}\, S_\bullet \underbrace{\setminus V(\varphi(T_+))}_{} \to \mathrm{Proj}\, T_\bullet$$

On points, this morphism is $[\mathfrak{p}] \mapsto [\varphi^{-1}(\mathfrak{p})]$.

(will have: $(\varphi^*)^{-1}(D(f)) = D(\varphi(f))$
$f \in T$

(Recall: $\mathrm{Proj}\, T = \{ [\mathfrak{p}] \in \mathrm{Spec}\, T \mid \begin{array}{l} \mathfrak{p} \text{ is homog.} \\ \mathfrak{p} \not\supseteq T_+ \end{array} \})$)

So if $V(\varphi(T_+)) = \emptyset$, get a morphism

$\varphi^*: \operatorname{Proj} S \to \operatorname{Proj} T$

$$\Updownarrow$$

$$\sqrt{\varphi(T_+)} = S_+$$

$$\boxed{\begin{array}{c} V(x_0^2, \dots, x_n^2) \subseteq \mathbb{P}^n \\ \parallel \\ \emptyset \end{array}}$$

<u>Examples:</u>

1) $\mathbb{P}^1_k \to \mathbb{P}^1_k$

$[x_0, x_1] \mapsto [f(x_0,x_1) : g(x_0,x_1)]$ relatively prime

2) $\varphi: S \to S/I$, I a homog. ideal defines $\varphi^*: \operatorname{Proj} S/I \to \operatorname{Proj} S$.

<u>Easy check:</u> this φ^* is a closed embedding.

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2 natural questions:

1) Can two different ideals $I_1, I_2 \subseteq S$ give the same closed subscheme? $\boxed{\text{Yes.}}$

2) Is every closed subscheme of $\operatorname{Proj} S$ given in this way? $\boxed{\text{Yes for reasonable } S.}$

First question: can show that

I, IS_+ define the same closed subscheme of Proj S.

Why? As sets, $\operatorname{Proj} S/I = \{[\mathfrak{p}] \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq I\}$

$$\operatorname{Proj} S/I = \underbrace{\{[\mathfrak{p}] \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq I\}}_{= V(I)}$$

A prime ideal that contains IS_+ but does not contain S_+ must contain I, so as sets, I, IS_+ cut out the same thing. As schemes, need to check that

$$\big(\underbrace{(S/IS_+)}\underbrace{[\tfrac{1}{f}]}\big)_0 \xrightarrow{\sim} \big((S/I)[\tfrac{1}{f}]\big)_0$$

for every homog. $f \in S_+$

So any element of I is annihilated by f in, so vanishes in the localization inverting f

Example: $\operatorname{Proj} k[x_0, x_1]/(x_1) \cong \operatorname{Proj} k[x_0, x_1]/(x_0x_1, x_1^2)$

$$\operatorname{Proj} k[x_0,x_1]/(x_1) = V(x_1) = \text{"origin" in } \mathbb{P}^1 \qquad \operatorname{Proj} k[x_0,x_1]/(x_0x_1,x_1^2) = V(x_0x_1, x_1^2)$$

Thm: Suppose $S_\bullet$ is a f.g. graded ring. Then every closed subscheme of Proj S is of the form $V(I) = \text{Proj}(S/I) \hookrightarrow \text{Proj } S$ for some (not unique) homog. ideal I.

Pf: (pset).

Defn: A *projective A-scheme* is an A-scheme isom to Proj $S_\bullet$ for $S_\bullet$ a f.g. graded ring over A ($S_0 = A$). // A *quasiprojective A-scheme* is a quasicompact open subscheme of a projective A-scheme.

immediate if A is Noetherian

think of a scheme "between $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$"

(Note: a somewhat tricky result gives that a proj. A-scheme is actually Proj $S_\bullet$ for $S_\bullet$ f.g. in degree 1, i.e. Proj $A[x_0,\dots,x_m]/I$, i.e. a closed subscheme of $\mathbb{P}^n_A$.)

Basic properties: (quasi) projective A-schemes are finite type (over Spec A) and qcqs

Why? we have a good understanding of an affine open base for Proj S, namely we have the $D(f)$ and they behave well, e.g.

$$D(f) \cap D(g) = D(fg)$$ is still true.

Summary:

- projective k-schemes are a large, well-behaved class of k-scheme, usually not affine.

- they look like $\mathrm{Proj}\ k[x_0,\dots,x_n]/(f_1,\dots,f_m)$, the $\underbrace{\text{closed k-valued}}_{\text{"classical"}}$ points are

$$\left\{ [x_0:\dots:x_n] \;\middle|\; \begin{array}{l} x_i \in k, \text{ not all } 0 \\ f_j(x_0,\dots,x_n)=0 \\ \text{for } j=1,\dots,m \end{array} \right\} \Big/ \underset{\text{scaling.}}{\sim}$$

- Many (but not all) morphisms

$$\mathrm{Proj}\ k[x_0,\dots,x_n]/(f_1,\dots,f_m) \longrightarrow \mathrm{Proj}\ k[y_0,\dots,y_{n'}]/(g_1,\dots,g_{m'})$$

are given by homog. polys compatible with (f_j), (g_j) that don't simultaneously vanish, i.e

such that $[x_0:\dots:x_n] \longmapsto [h_0(x_0,\dots,x_n):\dots:h_{n'}(x_0,\dots,x_n)]$

makes sense

($\mathrm{Proj}\ S \longrightarrow \mathrm{Spec}\ S_0$ is proper)

- more good properties to come:

"proper" ("complete") } "like being a compact manifold"
i.e. $\mathbb{P}^1_{\mathbb{C}} = \overline{\mathbb{C}}$ is compact, $\mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$ is not.

Def: A *hypersurface of degree d* in $\mathbb{P}^n_k$ is a closed subscheme defined by a single homog. polynomial of degree d, i.e.

$$V(f) \hookrightarrow \mathbb{P}^n_k \text{ for } f \in S_d \quad \left(S = k[x_0, \dots, x_n]\right)$$

(names for $d = 1, 2, 3, \dots$: hyperplane, quadric/conic, cubic, ...)

(e.g. $V(x_1^2)$ is a quadric hypersurface in $\mathbb{P}^1_k$
looks like "double point" at the "origin" $[1:0]$)

We know that given two closed subschemes $Z_1, Z_2 \hookrightarrow X$, we can get a third as the intersection.

Can check: $V(f) \cap V(g) = V(f, g)$

$$V(f,g) = \operatorname{Proj} k[x_0, \dots, x_n]/(f, g)$$

__Question:__ If we take n hypersurfaces of degrees $d_1, \dots, d_n$ inside $\mathbb{P}^n_k$, how many intersection points do we expect?

Bezout's Thm: Let k be a field. Let $n>0$. Suppose $f_1, \ldots, f_n \in k[x_0, \ldots, x_n]$ are homog. polynomials of degrees $d_1, \ldots, d_n$. Let $X = V(f_1) \cap \cdots \cap V(f_n)$ (scheme-theoretic intersection of hypersurfaces in $\mathbb{P}^n_k$), so

$$X \hookrightarrow \mathbb{P}^n_k \text{ is a closed subscheme.}$$
$$X = \operatorname{Proj} k[x_0, \ldots, x_n]/(f_1, \ldots, f_n)$$

Suppose that X is a finite k-scheme. Then $\mathcal{O}_X(X)$ is a k-vector space of rank exactly $d_1 d_2 \cdots d_n$.

$$= \sum_{p \in X} \text{"multiplicity at } p\text{"}$$