Last time: closed subschemes

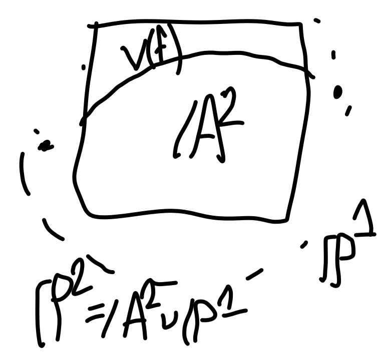
Question: What are the closed subschemes of $\mathbb{P}^n_k$?

Today: Proj construction (section 4.5)

Is every closed subset $Z \subseteq$ scheme $X$ the underlying set of some closed subscheme (maybe more than one)?

(Answer: yes)

We know $\mathbb{P}^n_k$ has lots of interesting closed subsets (because $\mathbb{A}^n_k$ does)



## Motivation:

A closed subscheme is locally cut out by ideals of functions (closed subschemes of Spec $A$ $\longleftrightarrow$ ideals $I \subseteq A$)

What are the functions on (open subsets of) $\mathbb{P}^n_k$?

Recall: $k$-valued points of $\mathbb{P}^n_k$ are

$$\left\{ [x_0 : \cdots : x_n] \;\middle|\; \begin{array}{l} x_i \in k \\ \text{not all } 0 \end{array} \right\} \Big/ \sim$$

scaling by $k^*$.

A polynomial $f \in k[x_0, \ldots, x_n]$ doesn't have well-defined values on such points,

but:

1) If $f, g$ are homog. polynomials of the same deg, then

$$\frac{f(x_0, \ldots, x_n)}{g(x_0, \ldots, x_n)} = \frac{f(cx_0, \ldots, cx_n)}{g(cx_0, \ldots, cx_n)} \in k \text{ makes sense}$$

whenever $g \neq 0$.

2) if $f$ is a homog. poly, then it makes sense to ask whether $f$ vanishes at a point $[x_0 : \cdots : x_n]$

Expectation: closed subschemes of $\mathbb{P}^n_k$ correspond to ideals generated by homog. polynomials.

(really ideals gen by $\dfrac{f}{\text{homog}}$ ← homogeneous on opens where denom doesn't vanish),

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Preview: just as closed subschemes of $\mathbb{A}^n_k$ looked like $\operatorname{Spec} k[x_1, \dots, x_n]/I$, closed subschemes of $\mathbb{P}^n_k$ will look like

$$\boxed{\operatorname{Proj}}\ k[x_0, \dots, x_n]/I$$

$I$ ← gen by homog. elements

$V(I) = V(\sqrt{I})$ as closed subsets but not as closed subschemes

$V(x^2) = (0)$

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Warning: multiple ideals $I$ can give the same closed subscheme.

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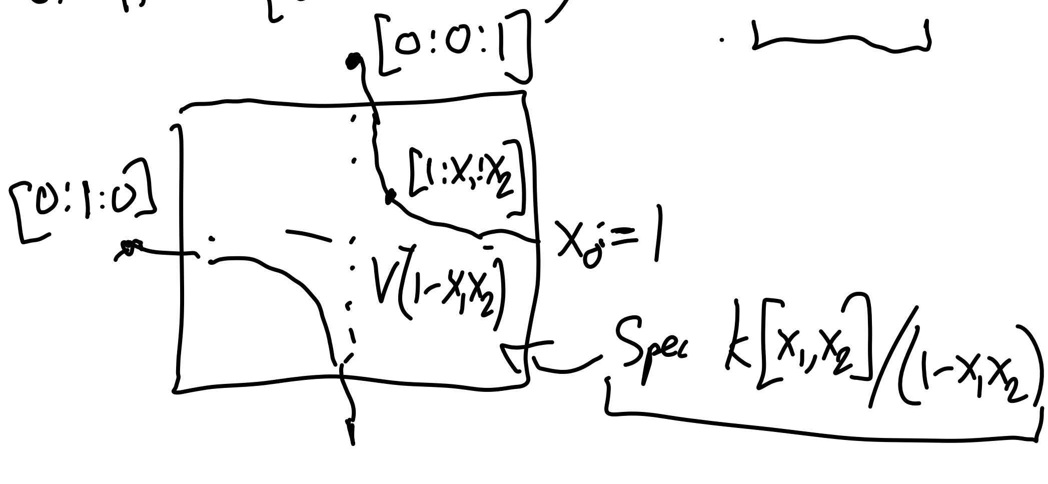
Example: The subset of $\mathbb{P}^2 = \{[x_0 : x_1 : x_2]\}$ where $x_0^2 - x_1 x_2 = 0$ will look like



## Graded rings:

**Notation/Defs:** A **$\mathbb{Z}$-graded ring** is a ring $S = S_\bullet$ along with an additive decomposition
$$S = \bigoplus_{n \in \mathbb{Z}} S_n \ (\text{where } S_n \subseteq S) \text{ s.t.}$$
"multiplication respects grading", i.e. have $m: S_m \times S_n \to S_{m+n}$. An element $f \in S_n$ is **homogeneous of degree $n$** and an ideal which can be generated by homog. elements is a **homog. ideal**.

**Convention:** A **graded ring** is a $\mathbb{Z}$-graded ring $S$ with $S_n = 0$ for $n < 0$. We say that $S$ is a **graded ring over $A$** if $A = S_0$.

Basic examples and properties:

- $S = A[x_0, \dots, x_n]$ is naturally $\overbrace{\text{a graded ring}/A}^{S_0 = A}$ if we assume $x_0, \dots, x_n \in S_1$.

- $S_0$ is a subring of $S$.

- If $I$ is a homogeneous ideal, $I$ has a compatible decomp $I = \bigoplus_{n \in \mathbb{Z}} I_n$ for $I_n = I \cap S_n$.

  - So $S/I$ has a natural induced grading.

[ $f \in k[x_1, \dots, x_n]$, not homog, deg $d$ $\quad (1 - x_1 x_2 \rightsquigarrow x_0^2 - x_1 x_2)$

$\rightsquigarrow \tilde{f} \in k[x_0, \dots, x_n]$ homog. of deg $d$ with

$\tilde{f}(x_0 = 1, x_1, \dots, x_n) = f(x_1, \dots, x_n)$ ]

(we will not be interested in non-homogeneous elts/ideals inside graded rings $S$.)

- If $f$ is a homog. element, $S[\frac{1}{f}]$ has an induced $\mathbb{Z}$-grading.

<u>Def</u>: Let $S$ be a graded ring. Then let
$$S_+ := \bigoplus_{n>0} S_n$$ is an ideal in $S$ and
is called the <u>irrelevant ideal</u>.

<u>Lemma</u>: $S$ can be f.g. $S_0$-alg by homog. elements $\iff S_+$ is a f.g. ideal

<u>Def</u>: A graded ring $S$ is <u>generated in deg 1</u> if $S$ is gen. by $S_1$ as an $S_0$-algebra.

($S$ is <u>f.g. in deg 1 over $A$</u> $\iff$
$$S \cong A[\underbrace{x_0, \dots, x_n}_{\text{deg } 1}] / \underbrace{I}_{\text{homog. ideal}}$$ )

Def: Let $S$ be a graded ring. Then $\operatorname{Proj} S$ is the scheme defined as follows (as a set, then top. space, then scheme)

(remember: $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n]$)

as a set: $\operatorname{Proj} S := \left\{ \mathfrak{p} \subset S \;\middle|\; \begin{array}{l} \mathfrak{p} \text{ prime} \\ \mathfrak{p} \text{ is homog}, \underbrace{\mathfrak{p} \not\supseteq S_+} \end{array} \right\}$

can check for primeness for homog. elements

(in case $S = k[x_0, \dots, x_n]$, $\underset{(x_0, \dots, x_n)}{S_+}$ is maximal and corresponds to the origin in $\mathbb{A}^{n+1}$, but we remove the origin when constructing $\mathbb{P}^n$)

as a top. space:

Given a homog. ideal $I \subseteq S$, let

$$V(I) = \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq I \}.$$

(so $V(I) \cong \operatorname{Proj} S/I$ as sets)

Define the topology on $\operatorname{Proj} S$ by saying that the $V(I)$ are the closed subsets.

Given a homog. elt $f \in S$, let

$$D(f) = \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \not\ni f \}$$

Easy check: the $D(f)$ are an open base for the topology of $\operatorname{Proj} S$.

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as a scheme:

<u>Lemma:</u> There is a natural <u>homeomorphism</u>

$$\begin{array}{ccccc}
\operatorname{Proj} S \supseteq D(f) & \longrightarrow & & & \operatorname{Spec}\left( \left(S_\bullet\left[\tfrac{1}{f}\right]\right)_0 \right) \\
\cup & & & & \\
f \notin \mathfrak{p} & \longmapsto & \mathfrak{p}' & \longmapsto & \mathfrak{p}' \cap \underbrace{\left(S_\bullet\left[\tfrac{1}{f}\right]\right)_0} \\
\cap & & \cap & & \\
S_\bullet & & S_\bullet\left[\tfrac{1}{f}\right] & &
\end{array}$$

$$D(f) \supseteq D(g) = D(fg)$$

<u>Pf:</u> (pset).

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This defines a structure sheaf on $D(f)$ for each homog. $f_{\not\mathfrak{p}_0} \in S$, and <u>they glue together</u>.

(We could have defined Proj $S$ in one step via this final gluing:

$$\operatorname{Proj} S = \bigcup_{\substack{f \in S \\ \text{homog, non-zero}}} \operatorname{Spec}\left(\left(S\left[\tfrac{1}{f}\right]\right)_0\right)$$

check isoms on triple intersections.)

<u>Example:</u> $\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \dots, x_n]$. Our old definition was gluing $D(x_0), \dots, D(x_n)$, but now we have easy access to many other affine opens, e.g. $D(x_0^2 - x_1 x_2) = \left[\begin{array}{c}\text{complement of} \\ \text{hyperbola in } \mathbb{P}^2\end{array}\right]$

and $\mathcal{O}_{\mathbb{P}^2_k}(D(x_0^2 - x_1x_2)) =$

$$\left(k\left[x_0, x_1, x_2, \frac{1}{x_0^2 - x_1x_2}\right]\right)_0 \ni \frac{x_0^2}{x_0^2 - x_1x_2}$$

$$= \left\{ \frac{f}{g} \in k(x_0, x_1, x_2) \;\middle|\; \begin{array}{l} f, g \text{ are homog. of the same degree} \\ \text{and } g = (x_0^2 - x_1x_2)^m \text{ for some } m \end{array} \right\}$$