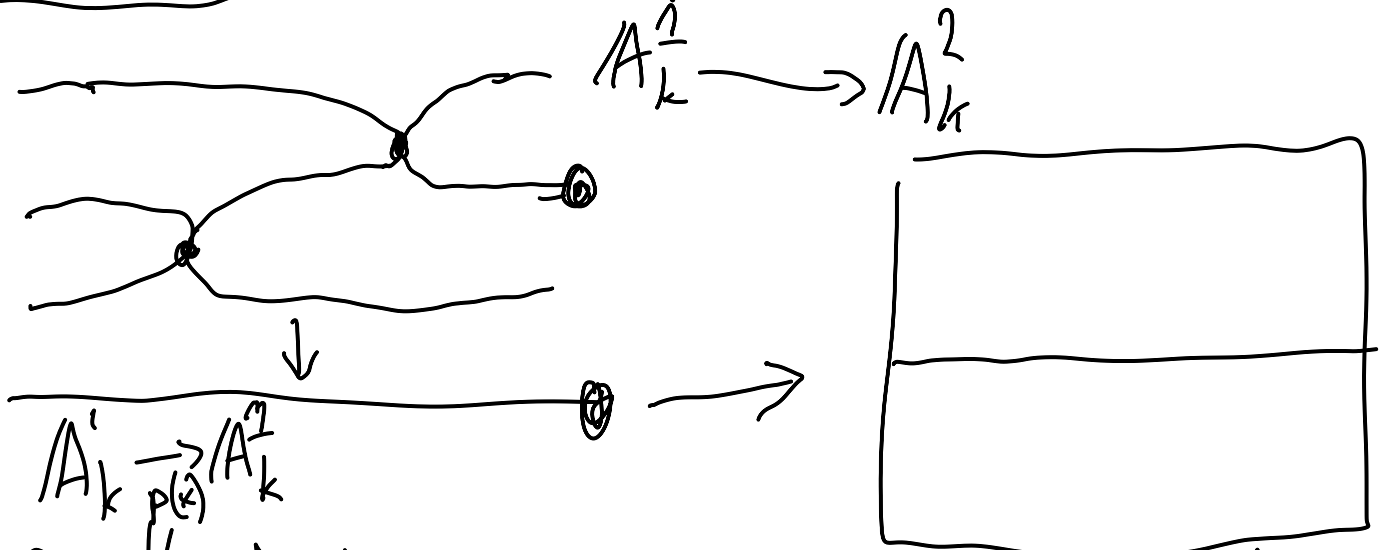
Last time: finite morphisms



Recall: finite k-schemes are finite sets with discrete topology

(not $\Longleftrightarrow$, e.g. $\operatorname{Spec} \mathbb{C}$ is not a finite $\mathbb{Q}$-alg)

Cor: (with a little work): finite morphisms have finite (set-theoretic) fibers.

pset

(idea:

$$\begin{tikzcd} \pi^{-1}(p) \arrow[r] \arrow[d, "\text{finite}"] & X \arrow[d, "\pi\text{, finite}"] \\ p \arrow[r, hook] & Y \end{tikzcd}$$

$p \simeq \operatorname{Spec} k_p$)

Def: A morphism $\pi: X \to Y$ is dominant if the image is dense.

(can check dominant on any open dense subset of X)

Note: If X, Y are integral, then π is dominant

$$\Longleftrightarrow \pi(\eta_X) = \eta_Y$$

generic points

Prop/Def: Let $\pi: X \to Y$ be a $\underbrace{\text{dominant finite}}_{\text{surjective finite}}$ morphism of integral schemes. Then the induced map on function fields $K(Y) \hookrightarrow K(X)$ is a finite extension of fields. The degree of π is the degree of this extension.

Pf: Check after restricting to affines:

$$\operatorname{Spec} A \to \operatorname{Spec} B$$

$B \to A$ is a finite extension of domains

Then need $K(B) \to K(A)$ is a finite extension of fields.

(comm. alg.) $\blacksquare$

Example: $\mathbb{A}^1_k \to \mathbb{A}^1_k$, $t \longmapsto p(t)$ has degree equal to the degree of p as a polynomial.

We'd now like to say: any finite morphism of integral schemes factors as

$$\begin{tikzcd}
X \arrow[rr, bend left, "{\pi,\ \text{finite}}"] \arrow[r, "\text{finite}", "\text{surjective}"'] & Z \arrow[r, "\text{finite}", "\text{injective}"'] & Y
\end{tikzcd}$$

has same degree

same "dimension"

, where "$Z = \text{image}(\pi)$" and $Z \to Y$ is a "<u>closed subscheme</u>"

On affines:

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[rr, bend left, "\pi"] \arrow[dr, "\text{surj by Lying Over.}"'] & & \operatorname{Spec} B \\
& \operatorname{Spec} B/\ker \pi^{\#} \arrow[ur] &
\end{tikzcd}$$

$\pi^{\#} : B \to A$

Def: A <u>closed embedding</u> $\pi : X \to Y$ is an affine morphism that looks like $\operatorname{Spec} A/I \to \operatorname{Spec} A$ over every open affine $\operatorname{Spec} A \subseteq Y$.

(A <u>closed subscheme</u> can be thought of as a closed embedding in which X is a subset of Y, or equiv. an isomorphism class of closed embedding into Y

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\text{isom}"'] & Y \\ X' \arrow[ur] & \end{tikzcd}$$

)

Easy: Any closed embedding is finite, injective
closed image.

Examples:

0) $\mathrm{Spec}\, A/I \to \mathrm{Spec}\, A$

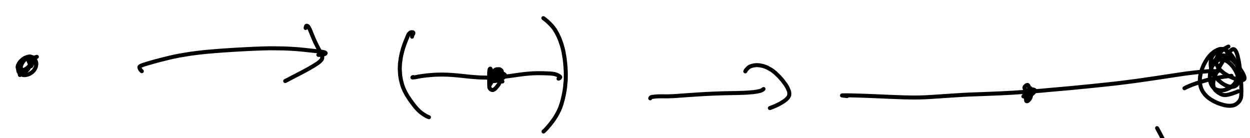
1) $p \hookrightarrow X$, $p \mapsto p$ if p is a closed point

$p = \mathrm{Spec}\, k_p$

[Why? Looks like $\mathrm{Spec}\, A/m_p \to \mathrm{Spec}\, A$ over open nhoods of p

and like $\mathrm{Spec}\, A/A \to \mathrm{Spec}\, A$ ($\mathrm{Spec}\, A/A = \emptyset$) over open sets not containing p.]

2) $\mathrm{Spec}\, k[t]/t \to \mathrm{Spec}\, k[t]/t^2 \to \mathrm{Spec}\, k[t]$



(closed subschemes of $\mathrm{Spec}\, A \longleftrightarrow$ ideals in A)

3) $\mathbb{P}^{n-1}_k \longrightarrow \mathbb{P}^n_k$

$[x_0 : \cdots : x_{n-1}] \longmapsto [x_0 : \cdots : x_{n-1} : 0]$

(over our standard $n+1$ affine opens in $\mathbb{P}^n_k$, this is either $\mathbb{A}^{n-1}_k \to \mathbb{A}^n_k$ or $\emptyset \to \mathbb{A}^n_k$)

<u>Question:</u> What are all the closed subschemes of $\mathbb{P}^n_k$ (for some n)?

(Ans: "projective k-scheme") ← special type of finite type k-scheme.

Tuesday

Analogous question for $\mathbb{A}^n_k$: $\underbrace{\operatorname{Spec} k[t_1, \ldots, t_n]/I}$, i.e.

Spec A with A a f.g. k-alg, i.e.
an arbitrary affine finite type k-scheme.

Two ways to think about the data of a closed subscheme $\pi : X \hookrightarrow Y$:

1) For each affine open $\operatorname{Spec} A \subseteq Y$, we have an ideal $I(A) \subseteq A$ defining $X \cap \operatorname{Spec} A$.

2) Consider the pullback map $\pi^\# : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ (of sheaves of rings on Y) and take its kernel open set by open set
$$\left[(\ker \pi^\#)(U) = \ker\left(\mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))\right) \right]$$
to define a new sheaf on Y, $\mathcal{I} := \ker \pi^\#$, "sheaf of functions on Y that vanish when pulled back to X"

"sub-$\mathcal{O}_Y$-module of $\mathcal{O}_Y$"

Here $\mathcal{I}$ is an "$\underline{\text{ideal sheaf}}$ of $\mathcal{O}_Y$" in the sense that $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_Y(U)$, with compatible restriction maps.

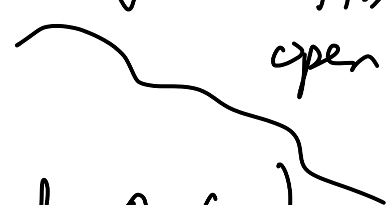
Thm: Fix a scheme Y. The following are equivalent (exist natural bijections between data):

1) A closed subscheme of Y

2) A choice of ideals $I_U \subseteq \mathcal{O}_Y(U)$ for all affine open U, s.t.

$$I_{D_U(f)} = I_U\left[\tfrac{1}{f}\right] \text{ for } f \in \mathcal{O}_Y(U)$$

$$\left(\begin{array}{ccc} \operatorname{Spec} A/I & \hookrightarrow & \operatorname{Spec} A \\ \cup\!\!\mid & & \cup\!\!\mid \\ \operatorname{Spec} A[\tfrac{1}{f}]/I[\tfrac{1}{f}] & \hookrightarrow & \operatorname{Spec} A[\tfrac{1}{f}] \end{array}\right)$$


U is an affine open

3) An ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ s.t.

$$\underbrace{\mathcal{I}(D_U(f)) = \mathcal{I}(U)\left[\tfrac{1}{f}\right]}_{\text{"}\mathcal{I}\text{ is quasicoherent"}} \text{ for } f \in \mathcal{O}_Y(U).$$

Key idea: use the result we used in the proof of the Affine Communication lemma to cover the intersection of any two affine opens with affine opens simultaneously distinguished in both of them.

(Note: given any morphism $X \xrightarrow{\pi} Y$, we consider the ideal sheaf

$$\mathcal{I} = \ker(\pi^\# : \mathcal{O}_Y \to \pi_* \mathcal{O}_X).$$

This will be quasicoherent, so $\mathcal{I}$ corresponds to some closed subscheme of Y. This will be the "scheme-theoretic image" of π.)

Examples/defs:

1) If X is a scheme and $f \in \mathcal{O}_X(X)$, we can define the <u>vanishing scheme</u> of f, denoted $V(f)$, as the closed subscheme of X given by the principal ideal gen by the image of f in each $\mathcal{O}_X(U)$.

Example: $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$

closed subschemes $V(t)$, $V(t^2)$

2) a) Given two closed subschemes $Z_1, Z_2 \subseteq X$, we can define their <u>union</u> to be the closed subscheme corresp to $\mathcal{I}_1 \cap \mathcal{I}_2$, where $\mathcal{I}_1, \mathcal{I}_2$ are the ideal sheaves corresp. to Z_1, Z_2.

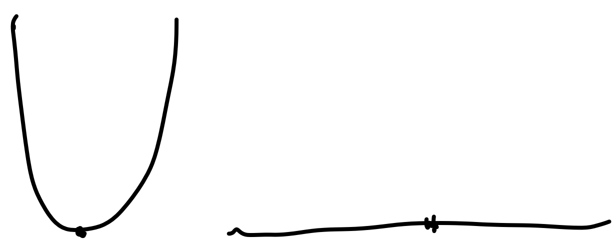
b) Similarly, given closed subschemes $\{Z_i\}$ in X, we can define their intersection by taking $\mathcal{I}$ = ideal sheaf spanned by all the $\mathcal{I}_i$.

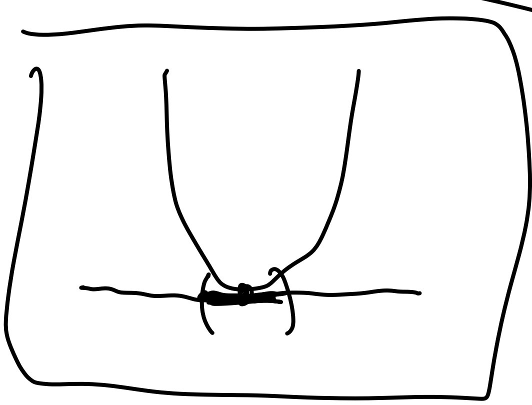
Example: In $\mathbb{A}^2_k$:

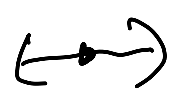
$$V(y-x^2) \cap V(y) = \operatorname{Spec} k[x,y]/(y-x^2, y)$$

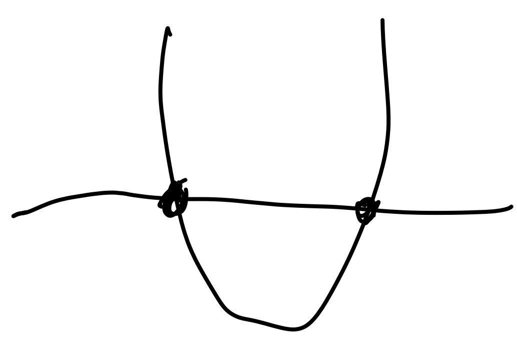
$$\cong \operatorname{Spec} k[x,y]/(y, x^2)$$

$$\cong \operatorname{Spec} k[t]/t^2$$









Example: $V(y-x^2) \cap V(y-1)$ is reduced: $\operatorname{Spec} k \sqcup \operatorname{Spec} k$