Thanks for the feedback you gave last Tuesday

- one change already made: list approximate topics/sections (in Vakil) ahead of time (posted on website below the psets — usually on Thursday or Friday)

This week: more with affine morphisms (7.3.5–7.3.11)
closed embeddings (8.1)

- we've been doing a lot of definitions/formalisms (might want to glance over some starred sections in Vakil for more payoff).

Recall: π is affine $\iff \pi^{-1}(U)$ is affine for all affine open U.

Prop: Affineness is affine-local on the target. ($P=$ "$\pi^{-1}(U)$ is affine" is an affine-local property)

Main ingredient in proof:

Qcqs Lemma: If X is qcqs and $s \in \mathcal{O}_X(X)$, then there is a natural isomorphism

$$\mathcal{O}_X(X)\left[\tfrac{1}{s}\right] \xrightarrow{\sim} \mathcal{O}_X(X_s),$$ where

$$X_s = \{p \in X \mid s(p) \neq 0\}$$ ($= D(s)$, though note X not assumed to be affine)

(Note: true for X affine by construction of $\mathcal{O}_{\operatorname{Spec} A}$)

Pf: What is this map? The restriction map $\mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(X_s)$ maps s to a unit, so factors uniquely through $\mathcal{O}_X(X)\left[\tfrac{1}{s}\right]$.

What does it mean that X is qcqs?

Can cover X with finitely many affine opens $U_i \cong \operatorname{Spec} A_i$ and then cover each $U_i \cap U_j$ with finitely many $U_{ijk} \cong \operatorname{Spec} A_{ijk}$.

Then the sheaf axioms say that

$$\mathcal{O}_X(X) \cong \left\{ (s_i) \in \prod_i \mathcal{O}_X(U_i) \;\middle|\; \begin{array}{c} s_i|_{U_{ijk}} = s_j|_{U_{ijk}} \\ \text{for all } i,j,k \end{array} \right\}$$

$$\cong \ker\left(\prod_i A_i \xrightarrow{(r_{U_i,U_{ijk}} - r_{U_j,U_{ijk}})} \prod_{i,j,k} A_{ijk} \right)$$

$\mathcal{O}_X(X)$-module homomorphism

Then $\mathcal{O}_X(X)[\frac{1}{s}] \cong \ker\left(\prod_i A_i \longrightarrow \prod_{ijk} A_{ijk}\right)[\frac{1}{s}]$

$$\cong \ker\left(\prod_i A_i[\tfrac{1}{s}] \longrightarrow \prod_{ijk} A_{ijk}[\tfrac{1}{s}]\right)$$

$\cong \mathcal{O}_X(X_s)$ do same computation with $X_s = \bigcup (U_i)_s$

localization commutes with kernels and finite products

$\blacksquare$

Pf of Prop (about affineness)

One hard thing to check:

If $\pi: X \to \operatorname{Spec} B$ is a morphism,

$(b_1, \dots, b_n) = (1) = B$, and

$\pi^{-1}(D(b_i)) \cong \operatorname{Spec} A_i$ for each i, then

we want to show that X is affine.

Initial observation: $\underbrace{\pi \text{ is a qcqs morphism}}$ (since $\pi^{-1}(D(b_i))$ is affine, hence qcqs)

so $\underset{\pi^{-1}(\operatorname{Spec} B)}{\overset{\shortparallel}{X}}$ is qcqs (as an object) since qcqs is affine-local on the target

Let $A = \mathcal{O}_X(X)$. There is a natural map

$\alpha: X \to \operatorname{Spec} A$ (corresp. to $\mathrm{id}_A : A \to \mathcal{O}_X(X)$).

We want to show α is an isomorphism.

It suffices to check that α is "covered by isomorphisms"

$\boxed{A = \mathcal{O}_X(X)}$

$$\begin{tikzcd}
X_{\varphi(b_i)} \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong" rotate=90] & \pi^{-1}(D(b_i)) \arrow[rr, hook] \arrow[dr, "\alpha'"] \arrow[dd] & & X \arrow[dr, "\alpha"] \arrow[dd, "\pi"] & \\
\operatorname{Spec} A_i & & D(\varphi(b_i)) \arrow[rr, hook] \arrow[dl] & & \operatorname{Spec} A \arrow[dl] \\
& D(b_i) \arrow[rr, hook] & & \operatorname{Spec} B &
\end{tikzcd}$$

corresp to induced map of π on global sections $= \varphi : B \to A$

Enough to show that α' is an isom.

Apply qcqs lemma to X and $\varphi(b_i) \in A = \mathcal{O}_X(X)$:

$$\mathcal{O}_X(X)\left[\frac{1}{\varphi(b_i)}\right] \cong \mathcal{O}_X\left(X_{\varphi(b_i)}\right)$$

So α' induces an isom on global sections

Then done since both $D(\varphi(b_i))$ and $\operatorname{Spec} A_i$ are affine. $\blacksquare$

<u>Cor</u>: Spec $A \to$ Spec B is affine;

$X \to$ Spec $\mathbb{Z}$ is affine $\iff$ X is affine.

<u>Cor 2</u>: (7.3.F, on next pset)

<u>Recall</u>: π is <u>integral</u> $\iff$ π is affine and over each affine open looks like Spec $A \to$ Spec B with A an integral B-alg, i.e. every element of A satisfies a monic poly. with coeffs in B.

(Can check affine-locality as normal).

Prop: ("Lying Over"): If $\varphi: B \to A$ is $\underbrace{\text{integral and injective}}_{\text{"integral extensions"}}$,

then $\varphi^*: \operatorname{Spec} A \to \operatorname{Spec} B$ is surjective.

$$\begin{array}{c} \exists q \subset A \\ \uparrow \varphi \\ p \subset B \\ \| \\ q \cap B \end{array}$$

Pf: (commutative algebra)

Cor: Integral morphisms of schemes are closed, i.e. the image of a closed set is closed.

Pf: Idea is on affines we have

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B \\ \operatorname{Spec} A/I \arrow[u] \arrow[r, "\text{surj.}"] & \operatorname{Spec} B/J \arrow[u] \end{tikzcd}$$

Given I, can choose J s.t.

and $B/J \to A/I$ is integral and injective. $\blacksquare$

Recall: Any finite morphism is integral, hence closed.

π is finite $\iff$ π is affine and looks like $\operatorname{Spec} A \to \operatorname{Spec} B$ with $\underbrace{A \text{ a f.g. } B\text{-module}}_{}$ over each affine open $\operatorname{spec} B$.

"A is a finite B-alg".

finite algebras are integral.

"finite k-scheme"

1) If $X \longrightarrow \operatorname{Spec} k$ is finite, then
$X \cong \operatorname{Spec} A$ with A a finite k-alg.

With some work, can show that this means

- X is a finite set.
- topology on X is discrete ("dimension 0")

- but the points can still h_ interesting structure:

$k = \mathbb{R}$: some finite $\mathbb{R}$-schemes:

$\operatorname{Spec} \mathbb{R}$

$\operatorname{Spec} \mathbb{C}$ "larger point"

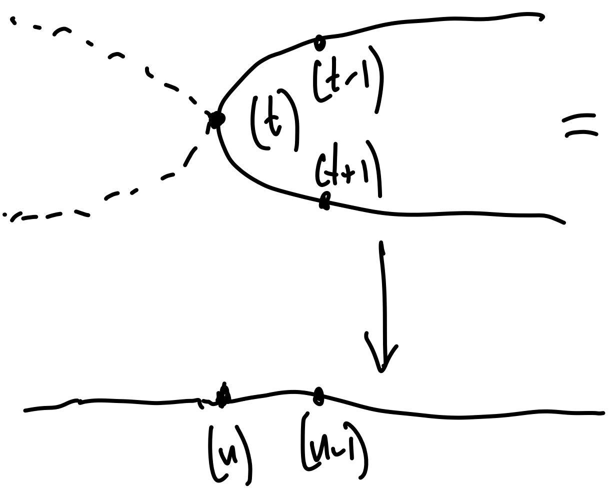
$\operatorname{Spec} \mathbb{R}[t]/t^2$ "fuzzy point"

2) $\mathbb{A}^1_k \xrightarrow{z^2} \mathbb{A}^1_k$ is finite, or more generally

$\mathbb{A}^1_k \xrightarrow{p(z)} \mathbb{A}^1_k$ is finite for any nonconstant polynomial $p \in k[z]$

$$\begin{array}{ccc} k[z] & \longleftarrow & k[x] \\ p(z) & \longmapsfrom & x \end{array}$$

finite because $k[z]$ is gen as $k[x]$-module by $1, z, \ldots, z^{d-1}$ for $d = \deg p$.



$= V(y - x^2)$

observation: fibers here are finite sets.

3) If $U \overset{\iota}{\hookrightarrow} X$ is an open subscheme, then ι is not finite unless U is also closed in X.

($A[\frac{1}{f}]$ is usually not f.g. as an A-module).

$\operatorname{Spec} \overline{\mathbb{Q}} \longrightarrow \operatorname{Spec} \mathbb{Q}$ is integral but not finite.