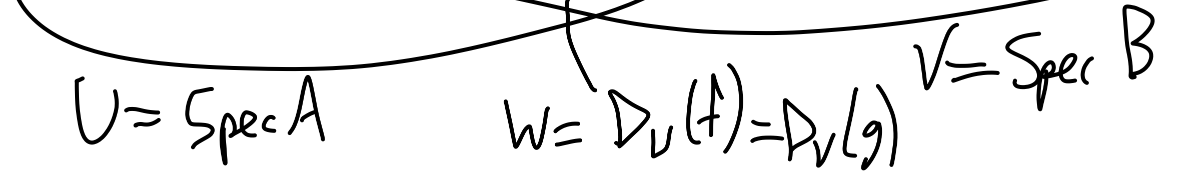
Today: how to check whether it is sufficient to check a property on an affine open cover?

Recall: $\pi: X \to Y$ is loc. finite type if for any affine opens $V \subseteq Y$ and $U \subseteq \pi^{-1}(V)$, $\mathcal{O}_X(U)$ is a f.g. $\mathcal{O}_Y(V)$-algebra.

Prop: Let U, V be two affine opens in the same scheme X. Then the intersection $U \cap V$ can be covered by affine opens W_i that are simultaneously distinguished opens in both U and V.

Given $p \in U \cap V$, want:



<u>Pf:</u> Choose $f' \in A$ s.t. $p \in D_U(f') \subseteq U \cap V$.

Then choose $g \in B$ s.t. $p \in D_V(g) \subseteq D_U(f')$.

(Now want $f \in A$ with $D_U(f) = D_V(g)$)

There is a restriction map

$$\begin{array}{ccc} B \cong \mathcal{O}_X(V) & \longrightarrow & \mathcal{O}_X(D_U(f')) \cong A\left[\frac{1}{f'}\right] \\ g & \longmapsto & h \end{array}$$

Let h be the image of g under this map.

Vanishing at a point is preserved by restriction,

so $D_V(g) = \{p \in V \mid g(p) \neq 0\}$

$= \{p \in D_U(f') \mid h(p) \neq 0\} = D_{D_U(f')}(h)$,

But $h \in A\left[\frac{1}{f'}\right]$ can be written as $\frac{h'}{(f')^n}$ for $h' \in A$.

Then $D_{D_U(f')}(h) = D_U(f'h')$,

So we may take $f = f'h'$. $\blacksquare$

Cor: (Affine Communication Lemma):

Let X be a scheme. Let P be a property possessed by some of the affine opens in X, such that

(a) If U has P and V is a distinguished affine open in U, then V has P

(b) If U is an affine open and $V_1, \dots, V_n$ are distinguished affine opens in U that all have P and cover U, then U has P.

$\operatorname{Spec} A = \bigcup_{i \in I} \operatorname{Spec} A[\frac{1}{f_i}]$

$\Leftrightarrow (f_1, \dots, f_n) = 1$

Then every affine open in X has P $\Longleftrightarrow$ X can be covered by affine opens that have P.

(Def: such a prop. P (satisfying (a) and (b)) is called affine-local)

Pf: Suppose $\{W_i\}_{i \in I}$ is a cover of X by affine opens with P, and suppose U is an arbitrary affine open.

Then $U = \bigcup_{i \in I} (U \cap W_i)$ can be covered by affine opens that are simultaneously distinguished in both U and some W_i.

Then use (a), then (b). ☒

Example: "the property of being a loc.-finite type A-scheme is affine-local"

Suppose X is an A-scheme. Say that an affine open $U \subseteq X$ has property P if $\mathcal{O}_X(U)$ is a f.g. A-algebra.

Claim: P is an affine-local property.

Pf: B f.g. A-alg $\Longrightarrow B[\frac{1}{f}]$ is a f.g. A-alg (add $\frac{1}{f}$ to the gens).

Other direction: Suppose B is an A-alg and $c_1, \ldots, c_n, f_1, \ldots, f_n \in B$ satisfy $\sum_i c_i f_i = 1$ (so $\bigcup_{i=1}^{n} D(f_i) = \operatorname{Spec} B$), and suppose $B[\frac{1}{f_i}]$ are f.g. A-algs for all i.

Then can check that the numerators for the gens of the $B[\frac{1}{f_i}]$ $(i = 1, \ldots, n)$ along with all the c_i and f_i suffices to generate B as an A-alg. $\blacksquare$

<u>Consequence:</u> Given an A-scheme X, we've proven that

$\mathcal{O}_X(U)$ is a f.g. A-alg for affine open $U \subseteq X$ covering X

$\iff \mathcal{O}_X(U)$ is a f.g. A-alg for all affine opens $U \subseteq X$.

<u>Example:</u> "the property of being a loc. finite type morphism is affine-local on the target."

Suppose $\pi: X \to Y$ is a morphism. Say that $U \subseteq Y$ has property P if $\pi^{-1}(U)$ is a loc. finite type $\mathcal{O}_Y(U)$-scheme. $(\pi : \pi^{-1}(U) \to U)$

<u>Pf that P is an affine-local property:</u>

(a) B f.g. A-alg $\Rightarrow B[\frac{1}{f}]$ is a f.g. $A[\frac{1}{f}]$-alg.

(b) B is a f.g. $A[\frac{1}{f}]$-alg $\Rightarrow B$ is a f.g. A-alg, and previous affine-localness for being a loc. finite type A-scheme.

Other examples of affine-local properties:

1) $U = \operatorname{Spec} A$ with A Noetherian
(so being loc. Noetherian can be checked on a cover)

2) $\pi^{-1}(U)$ is quasicompact (resp. quasiseparated)
(so being a qc (resp qs, qcqs) morphism can be checked on an affine open cover of the target)

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3 properties that any "nice" property of morphisms of schemes should satisfy:

1) "affine-local on the target"
~ so $\pi^{-1}(U) \to U$ having the property is affine-local.

2) closed under composition and id_X has the property.

3) closed under "base change" (= fiber product, will discuss more later)

fiber product $=$

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\text{should have } Q"] \arrow[d] & Z \arrow[d] \\ X \arrow[r, "\text{has } Q"'] & Y \end{tikzcd}$$

base change of

Def: A morphism $\pi : X \to Y$ is <u>affine</u> if $\pi^{-1}(U)$ is affine for every affine open $U \subseteq Y$.

Q: Is "$\pi^{-1}(U)$ is affine" an affine-local property on $U \subseteq Y$?

(Surprisingly tricky; but yes – will prove next week)

For now: unclear whether $\operatorname{Spec} A \to \operatorname{Spec} B$ is necessarily an affine morphism.

Example: $\operatorname{Spec} A \to \operatorname{Spec} k$ is an affine morphism;

$X \to \operatorname{Spec} k$ is an affine morphism $\Longleftrightarrow$ X is affine.

Affine morphisms: most interesting when Y is not affine.

Example: Any morphism $\underset{p}{\underset{\shortparallel}{\operatorname{Spec} k_p}} \hookrightarrow \underset{p}{\underset{\rotatebox{90}{$\in$}}{X}}$ is affine.

Can define various special classes of affine morphisms by saying that not only is

$$\pi^{-1}(\operatorname{Spec} A) \text{ isom to } \operatorname{Spec} B,$$

but also placing some condition on the ring homomorphism

$$A \longrightarrow B.$$

<u>Examples</u>:

ring homom. condition	subclass of affine morphism
surjective ($B \cong A/I$)	closed embedding.
	$\Downarrow$
B is a f.g. A-module	finite morphism.
	$\Downarrow$
B is integral over A - each elt $x \in B$ satisfies a monic poly with coeffs in the image of A	integral morphism. <u>Warning</u>: has nothing to do with integral schemes.

most geom interest

technical condition