Last time: various properties of schemes

Map of basic named properties of schemes

	naturally expressed in terms of affine opens (usually prop of morphisms)	more naturally expressed in some other way (usually ~~not~~ prop of morphisms)
topological	* quasicompact * quasiseparated	connected irreducible open embedding
ring-theoretic (correspond to props of rings or ring homoms)	* locally finite type * locally Noetherian affine $\left\{\begin{array}{l}\text{finite}\\ \text{integral}\\ \text{closed embedding}\end{array}\right.$	reduced, normal: conditions on $\mathcal{O}_{X,p}$ today

have defined these now

Recall: X is reduced $\iff \mathcal{O}_{X,p}$ is reduced for all $p \in X$

$\iff \mathcal{O}_X(U)$ is reduced for all open $U \subseteq X$

($\iff \mathcal{O}_X(U)$ is reduced for some affine open cover)

Def: X is <u>integral</u> if X is reduced and irreducible

reduced — "no fuzz"   irreducible — "one piece"

Examples: $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, $\operatorname{Spec} k[x_1, \ldots, x_n]/(f)$ for irreducible f.
($\operatorname{Spec} A$ for any domain A).

Prop/alt. def: X is integral $\iff$
$\mathcal{O}_X(U)$ is a domain for every nonempty open $U \subseteq X$.

Pf: $\Rightarrow$: Suppose for contradiction that $f, g \in \mathcal{O}_X(U)$ are nonzero but $fg = 0$. Then $V(f), V(g)$ are <u>proper</u> closed subsets of U $\leftarrow$ from reducedness of X

and $V(f) \cup V(g) = U$, so U is reducible, and hence so is X.

$\Leftarrow$ (assuming $\mathcal{O}_X(U)$ is a domain for all U):

Reducedness is immediate, since domains are reduced rings.

For irreducibility: suppose $U \cong \operatorname{Spec} A$ is an affine open in X, and let η be the generic point of U $\left(\begin{smallmatrix}\text{corresp to}\\ \text{zero ideal in } A\end{smallmatrix}\right)$.

Then if V is any nonempty open in X, we must have $U \cap V \neq \emptyset$ (since $\mathcal{O}_X(U \cup V)$ is a domain) but then $\eta \in U \cap V$ since η is dense in U, so $\eta \in V$ and we see that η is dense in X. $\blacksquare$

Def: Let X be an integral scheme. Then the <u>function field</u> of X, denoted $K(X)$ is the stalk of $\mathcal{O}_X$ at the generic point of X.

Elements $f \in K(X)$ are called <u>rational functions</u> on X.

Example: $K(\mathbb{A}^n_k) \cong k(x_1, \dots, x_n)$

$K(\mathbb{P}^n_k) \cong K(\mathbb{A}^n_k)$ $\left(\underbrace{\text{can compute stalk in any affine open}}\right)$.

Notes: 1) If $U \cong \operatorname{Spec} A$ is some nonempty affine open in X, then $K(X) \cong A_{(0)} =: K(A)$, the field of fractions of A.

2) All of the restriction maps

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V) \quad (\text{for } V \neq \emptyset)$$

are injective and combine to naturally identify $\mathcal{O}_X(U)$ with a subring of $K(X)$ for every nonempty open $U \subseteq X$.

(example: $X = \mathbb{P}^1_k$, $U = \mathbb{P}^1_k$: $\mathcal{O}_X(U) = k \subseteq k(t)$).

3) Any rat. function $f \in K(U)$ has a maximum "domain of definition", i.e. there is some open $U^{max} \subseteq X$ s.t.

$$f \in \mathcal{O}_X(U) \subseteq K(X) \iff U \subseteq U^{max}$$

Examples: the domain of definition of $\frac{x^2-1}{x-2} \in K(\mathbb{A}^1_k)$ is $\underbrace{\mathbb{A}^1_k - \{2\}}$ $= \operatorname{Spec}(k[x][\frac{1}{x-2}])$

(Remark: a restatement of some of the above is that there is a sheaf morphism embedding

$$\mathcal{O}_X \hookrightarrow \underline{K(X)}$$ ← constant sheaf.)

technical conditions we won't use

Other stalk-local conditions: <u>normal</u>, factorial, Cohen-Macaulay, regular

coming from putting same comm. alg. condition on the local rings $\mathcal{O}_{X,p}$

regular "nonsingular"

Later

<u>Def</u>: X is <u>normal at a point $p \in X$</u> if $\mathcal{O}_{X,p}$ is an integrally closed domain.

(A domain A is integrally closed (in its field of fractions) if: $\alpha \in A_{(0)}$ and $f \in A[x]$ is a <u>monic</u> polynomial with $f(\alpha) = 0$, then $\alpha \in A$.)

(Examples: $\mathbb{Z}$, $k[x_1, \ldots, x_n]$ are integrally closed)

<u>Normalization</u>: Given a scheme X, there is a universal modification of X that is normal: a normal scheme $\widetilde{X}$ and a morphism $\widetilde{X} \longrightarrow X$ with dense image.

<u>Idea of construction for X integral</u>: replace each affine open $\operatorname{Spec} A \subseteq X$ with $\operatorname{Spec} \widetilde{A} \subseteq \widetilde{X}$, where $\widetilde{A}$ is the integral closure of A inside $A_{(0)}$

Examples for "curves": idea is normal = "nonsingular" for curves

1) $X = \operatorname{Spec} k[t]$ normal already

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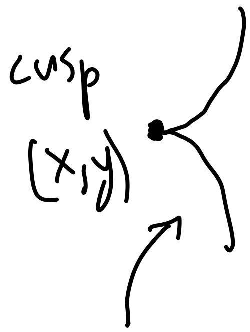
2) $X = \operatorname{Spec}\left(k[x,y]/(y^2 - x^3)\right)$: normal except at origin, normalization "fixes" the cusp:

$\cong$ $\operatorname{Spec} k[t^2, t^3]$



$\tilde{X} \cong \operatorname{Spec} k[t]$, morphism $\tilde{X} \longrightarrow X$ corresponds to ring homomorphism

$$k[x,y]/(y^2 - x^3) \to k[t]$$
$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

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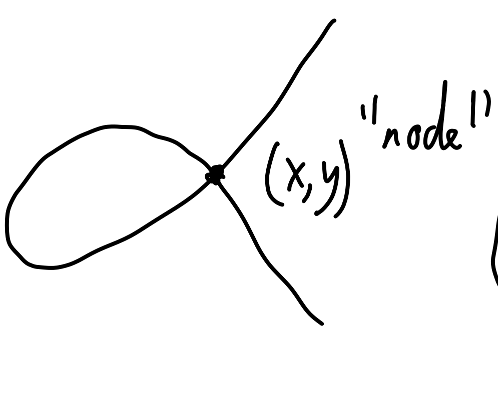
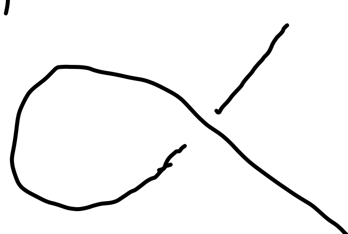
3) $X = \operatorname{Spec} k[x,y]/(y^2 - x^3 - x^2)$



normal except at origin, normalization separates out two branches



$\cong \mathbb{A}^1_k$ $\quad x = t^2 - 1$, $\; y = t(t^2 - 1)$