Last time: morphisms of loc. finite type:

$X \to Y$ whenever $\begin{matrix} \cup\!| & & \cup\!| \\ \operatorname{Spec} A & \to & \operatorname{Spec} B \end{matrix}$ $\rightsquigarrow B \to A$

A is a f.g. B-algebra

Today: start on properties of schemes

Schemes have various parts (top. space, structure sheaf)

- lots of useful, reasonable conditions to place on these parts.
- Vakil: chapter 5 (schemes)
  chapter 7 (morphisms)

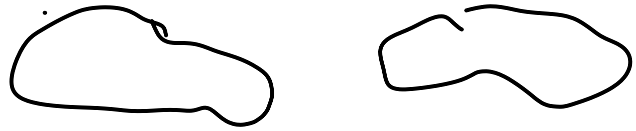
Map of basic named properties of schemes

	naturally expressed in terms of affine opens (usually prop of morphisms)	more naturally expressed in some other way (usually **not** prop of morphisms)
topological	* quasicompact * quasiseparated	connected irreducible open embedding
ring-theoretic (correspond to props of rings or ring homoms)	* Locally finite type * Locally Noetherian affine { finite, integral, closed embedding }	reduced normal } conditions on $\mathcal{O}_{X,p}$

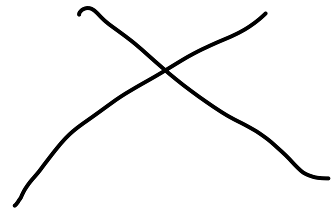
more complicated properties we'll get to later:

separated, proper, dimension d, regular, smooth ...

Connected, irreducible:



disc.



reducible

<u>Lemma</u>: The irred closed subsets of a scheme X are precisely the closures of points.

<u>Cor</u>: X is irred $\Longleftrightarrow$ X has a generic dense point η, i.e.

$$\overline{\{\eta\}} = X$$

<u>Def</u>: A morphism $\pi: X \to Y$ is an <u>open embedding</u> if it is isomorphic to an open subscheme, i.e. there is an open subset $U \subseteq Y$ and an isomorphism of schemes $X \xrightarrow{\sim} U$ s.t. π factors

$$X \xrightarrow{\sim} U \hookrightarrow Y.$$

Quasicompact (every open cover has a finite subcover):

Lemma/new def: A scheme X is quasicompact $\Longleftrightarrow$ X can be covered by finitely many affine opens.

Pf: $\Rightarrow$: affine opens cover X
$\Leftarrow$: Spec A is quasicompact. ▨.

Def: A morphism $\pi: X \to Y$ is quasicompact if $\pi^{-1}(U)$ is quasicompact for every affine open $U \subseteq Y$.

(Note: $id_X : X \to X$ is q.c. for every scheme X).

(Def: A morphism $\pi: X \to Y$ is of finite type if it is locally of finite type and quasicompact.)

Quasiseparated:

Def: X is q.s. if the intersection of any two quasicompact opens $U, V \subseteq X$ is quasicompact.

Lemma/new def: X is quasiseparated $\iff$ the intersection of any two affine opens U, V is a finite union of affine opens.

Notation: qcqs = quasicompact + quasiseparated.

Note: X is qcqs $\iff$ X can be covered by finitely many affine opens U_i and the pairwise intersections $U_i \cap U_j$ can also be covered by finitely affine opens

"can describe X with finitely many rings"

($\pi: X \to Y$ is quasiseparated if $\pi^{-1}(U)$ is for every affine open U.

We could talk about the property of being a Noetherian top. space, but instead we prefer:

Def: A scheme X is <u>Locally Noetherian</u> if $\mathcal{O}_X(U)$ is a Noetherian ring for every affine open $U \subseteq X$.

Def: <u>Noetherian</u> = Locally Noetherian + quasicompact.

Lemma: Noetherian schemes are Noetherian top. spaces.

Lemma: If Y is Noetherian and $\pi: X \to Y$ is finite type, then X is Noetherian.

Pf: Hilbert basis theorem. ∎

Cor: Finite type k-schemes are Noetherian.

(Note for qcqs: affine schemes are qcqs because $D(f) \cap D(g) = D(fg)$, so intersections of affine opens in Spec A will be finite unions of distinguished affine opens.)

Stalk-local properties:

Def: X is reduced at $p \in X$ if $\mathcal{O}_{X,p}$ is a reduced ring, i.e. has no nonzero nilpotents.

X is reduced if it is reduced at every point.

Lemma: X is reduced $\Leftrightarrow$ $\mathcal{O}_X(U)$ is reduced for all open $U \subseteq X$.

Pf: $\Rightarrow$: $\mathcal{O}_X(U) \hookrightarrow \prod_{p \in U} \mathcal{O}_{X,p}$ means $\mathcal{O}_X(U)$ is reduced.

$\Leftarrow$: A reduced $\Rightarrow$ A_p is reduced. (comm. alg.)

(Note: enough to check $\mathcal{O}_X(U)$ is reduced for U in an affine open cover)

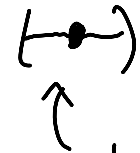
"philosophy": "functions on reduced schemes behave well":
or morphisms from

Lemma: If X is reduced and $f \in \mathcal{O}_X(U)$ vanishes at every point of U, then $f = 0$.

Pf: $\mathrm{Nil}(A) = \bigcap_{p \in \mathrm{Spec} A} p$ and identity axiom for sheaves. ☒

Some non-reduced schemes:

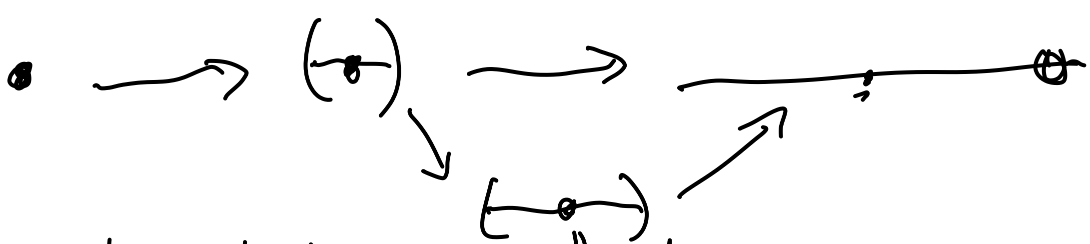
$$\operatorname{Spec} k[t]/(t^2)$$



single point with infinitesimal "fuzz" along $\operatorname{Spec} k[t]$

Morphisms:

$$\operatorname{Spec} k[t]/t \to \operatorname{Spec} k[t]/t^2 \longrightarrow \operatorname{Spec} k[t]$$



Look at pullback maps on functions:

$$k[t]/t \longleftarrow k[t]/t^2 \longleftarrow k[t]$$

$$k[t]/t^2 \longleftarrow k[t]/t^2 \longleftarrow k[t]$$

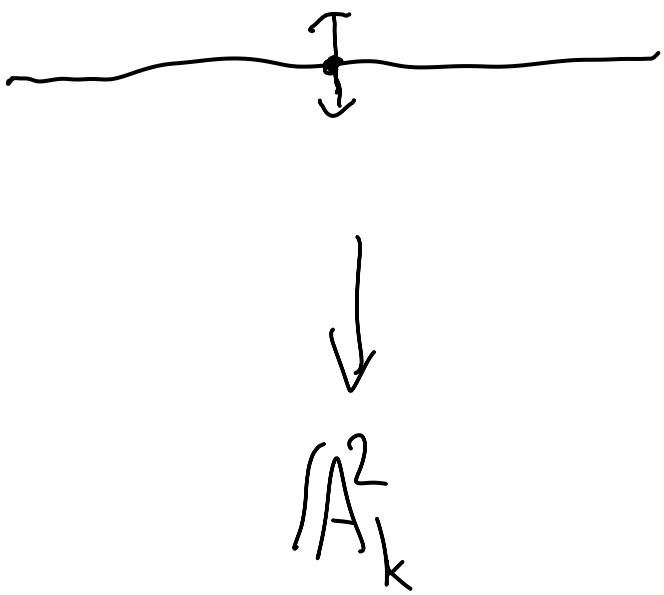
The reduced point just remembers the value of a function $f \in k[t]$ at $t=0$.

The fuzzy/nonreduced point also remembers the derivative at $t=0$.

Similarly: $\operatorname{Spec} k[t]/(t^3)$ has "more fuzz

More complicated example:

$\operatorname{Spec} k[x,y]/(y^2, xy)$



reduced except at origin, fuzz in direction orthogonal from line there.

<u>Def</u>: X is <u>integral</u> if X is reduced and irreducible.