Last time: morphisms of schemes:
~~on~~ affine opens, look like maps
Spec $A \to$ Spec B corresponding to
ring homomorphisms $B \to A$.

Today: - various things about morphisms
$X \to \operatorname{Spec} A$
and $\operatorname{Spec} A \to X$ (often $A = k$)

Simple (but important) example: Suppose L, K are fields.
Then Spec L, Spec K are single points, but
$\operatorname{Mor}_{Sch}(\operatorname{Spec} L, \operatorname{Spec} k)$ is interesting: it is
in bijection with $\operatorname{Mor}_{Rings}(K, L)$, the set
of field extension maps $K \hookrightarrow L$.

You should think of Spec K, Spec L as "points of different sizes" (for $K \neq L$).

<u>Example</u>: Spec $\mathbb{C}$ should be "twice as large" as Spec $\mathbb{R}$ (since $[\mathbb{C}:\mathbb{R}]=2$), and there are no morphisms Spec $\mathbb{R} \to$ Spec $\mathbb{C}$ but there is a "double cover"

$$\text{Spec } \mathbb{C} \longrightarrow \text{Spec } \mathbb{R}.$$

<u>Example</u>: Spec k often has nontrivial autom group corresponding to field auts of k.

Morphisms to affine schemes:

(Recall: we had $\mathbb{P}^n_k \to \operatorname{Spec} k$ defined by gluing $\mathbb{A}^n_k \to \operatorname{Spec} k$ (corresp. to $k \to k[t_1, \ldots, t_n]$))

<u>Lemma:</u> The morphisms $X \to \operatorname{Spec} A$ correspond to ring homomorphisms $A \to \mathcal{O}_X(X)$, i.e. A-alg. structures on the ring of global functions on X.

<u>Pf</u>: Given a morphism $\pi: X \to \operatorname{Spec} A$, the pullback map on global sections defines a map $A \to \mathcal{O}_X(X)$. In the other direction, $A \to \underbrace{\mathcal{O}_X(X) \xrightarrow[\text{res}]{} \mathcal{O}_X(U)}$ defines morphisms $U \to \operatorname{Spec} A$ for any affine open U, and these glue to give a morphism $X \to \operatorname{Spec} A$. These are inverses because they are for X affine. $\square$.

"$\pi: X \to \operatorname{Spec} A$ is the same thing as making everything in $\mathcal{O}_X$ an A-algebra".

Def: A scheme X equipped with a "structure morphism" $X \to \operatorname{Spec} A$ is called an A-scheme.

(alternatively: take defs of scheme and ringed space, etc, and replace "ring" with "A-algebra")

(**Note:** $\mathbb{Z}$-algebra $=$ ring, so "scheme $=$ $\mathbb{Z}$-scheme"
$\Leftrightarrow$ equiv, $\operatorname{Spec}\mathbb{Z}$ is the final object in Sch.)

Def: A morphism of A-schemes $X \to Y$ is a morphism of schemes commuting with the structure morphisms to $\operatorname{Spec} A$

$$
\begin{tikzcd}
X \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & Y \arrow[dl] \\
& \operatorname{Spec} A &
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
\mathcal{O}_X(X) & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{O}_Y(Y) \arrow[ll] \\
& A \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

There is only one morphism of $\mathbb{C}$-schemes $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$, but many morphisms of $\mathbb{Z}$-schemes.

Morphisms from Spec k:

<u>Lemma</u>: The morphisms $\pi: \operatorname{Spec} k \to X$, $* \mapsto p$ (with $p \in X$) correspond to injections of fields $k_p \hookrightarrow k$, where k_p is the residue field at p.

<u>Pf</u>: (special case of pset problem since k is a local ring)

In particular, there is a canonical map

$$\begin{array}{ccc} \operatorname{Spec} k_p & \longrightarrow & X \\ * & \longmapsto & p \end{array}$$

inducing the identity map on residue fields.

(explicitly: take an open affine $\operatorname{Spec} A \ni p$, $p = [\mathfrak{p}]$, and then

$$A \to A/\mathfrak{p} \to (A/\mathfrak{p})_0 \cong A_{\mathfrak{p}}/m_{\mathfrak{p}} \cong k_p.$$

We will sometimes refer to this morphism $\operatorname{Spec} k_p \to X$ as $p \hookrightarrow X$, $p = \operatorname{Spec} k_p$.

Note: Suppose $\pi : X \to Y$ is a morphism of schemes and $\pi(p) = q$. Then there is a commuting square

$$\begin{tikzcd} \operatorname{Spec} k_p = p \arrow[r, hook] \arrow[d] & X \arrow[d, "\pi"] \\ \operatorname{Spec} k_q = q \arrow[r, hook] & Y \end{tikzcd}$$

Def: Let A be a ring, and suppose that B is an A-algebra and X is an A-scheme. Then the set of B-valued points of X is

$$X(B) := \left\{ \text{morphisms of } A\text{-schemes } \operatorname{Spec} B \to X \quad \left(\begin{tikzcd}[column sep=small] \arrow[dr] & & \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd} \right) \right\}$$

(Generalization: can replace $\operatorname{Spec} B$ by an arbitrary A-scheme Z and take $X(Z) := \operatorname{Mor}_{A\text{-Sch}}(Z, X)$)

Then $Z \mapsto X(Z)$ defines a functor $A\text{-Sch} \to \text{Sets}$
"functor of points"

Justification of language:

Suppose $X = \operatorname{Spec} A[t_1, \dots, t_n]/(f_1, \dots, f_m)$

(Our picture of X: "closed subscheme" of $\mathbb{A}^n_A$ cut out by polynomial equations f_i)

Then $X(B) \cong \{$A-scheme morphisms $\operatorname{Spec} B \to X\}$

$\cong \{$A-alg homomorphisms $A[t_1, \dots, t_n]/(f_1, \dots, f_m) \to B\}$

$\cong$

$b_i :=$ image of t_i

$\{(b_1, \dots, b_n) \in B^n \mid f_i(b_1, \dots, b_n) = 0$ for $i = 1, \dots, m\}$ ($\in B$)

Example: $X = \operatorname{Spec} \mathbb{Q}[x,y,z]/(x^3+y^3-z^3)$:

$X(\mathbb{Q}) =$ (trivial solutions)

$X(\mathbb{C}) =$ (some hypersurface in $\mathbb{C}^3$)

Special case: The set of k-valued points of a k-scheme X is in bijection with the set of closed points of X with residue field exactly k.

"classical points"

Example: $\mathbb{A}^n_k(k) \cong k^n$.

Properties of schemes (and of morphisms of schemes):

(sometimes: instead of thinking about a scheme X having a property P, we should think about the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ or $X \to \operatorname{Spec} k$, etc having property P)

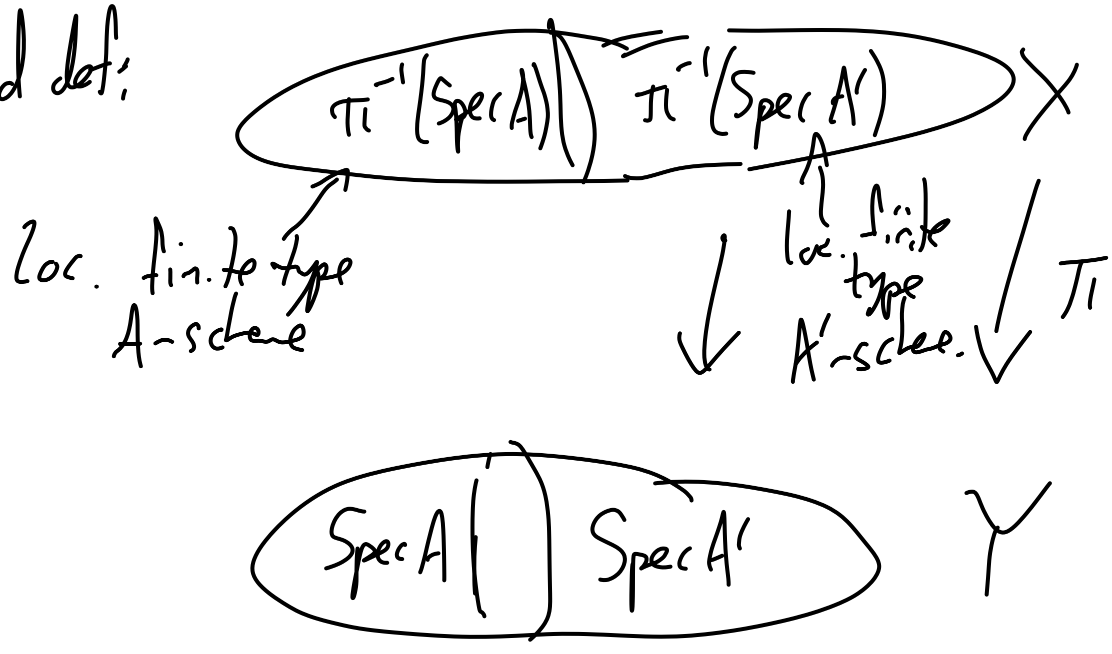
Def: Let X be an A-scheme. If $\mathcal{O}_X(U)$ is finitely generated as an A-algebra for every affine open $U \subseteq X$, we say that X is a $\underbrace{\text{locally}}$ finite type A-scheme.

(some sort of finite-dimensionality: $\mathcal{O}_X(U)$ looks like $A[t_1, \dots, t_n]/I$, so U looks like a closed subset of $\mathbb{A}^n_A$)

<u>Better def</u>: Let $\pi : X \to Y$ be a morphism. If $\mathcal{O}_X(U)$ is f.g. as an algebra over $\mathcal{O}_Y(V)$ for every pair of affine opens $U \subseteq X$, $V \subseteq Y$, with $\pi(U) \subseteq V$, then we say that π is <u>locally of finite type</u>.

(Immediate: X is a locally finite type k-scheme $\iff$ $X \to \operatorname{Spec} k$ is locally of finite type.)

Idea of second def:



Hard-ish result (which we will discuss later):

these two loc. finite type definitions (and several other similar defs)

can be checked on affine open covers, i.e.

if X is an A-scheme and $\mathcal{O}_X(U_i)$ is a f.g. A-algebra for $\{U_i\}$ some affine open cover of X, then X is a loc. finite type A-scheme.

Cor: If B is a f.g. A-algebra, then $\operatorname{Spec} B$ is a loc. finite type A-scheme.

Cor: X is a loc. finite type A-scheme $\Longleftrightarrow$ $X \to \operatorname{Spec} A$ is loc. of finite type.