<u>Last time</u>: gluing schemes, $\mathbb{P}^1_k$

<u>Today</u>: $\mathbb{P}^n_k$, morphisms of schemes

$\underbrace{\text{Classical}}_{\text{(closed points, } k=\bar{k})}$ perspective: $\mathbb{P}^n_k =$ " $\underbrace{\text{lines}}$ through origin in k^{n+1}"

replace with higher dim subspace to get Grassmannians

$$= \left(k^{n+1} - \{0\}\right)/k^*$$

$$= \left\{ [a_0 : \cdots : a_n] \;\middle|\; \begin{matrix} a_i \in k \\ \text{not all } 0 \end{matrix} \right\} \Big/ [a_0 : \cdots : a_n] \sim [ta_0 : \cdots : ta_n]$$

$$\left[= \mathbb{A}^n_k \cup \mathbb{P}^{n-1}_k \right.$$

$\mathbb{A}^n_k$: $a_0 \neq 0$, "$a_0 = 1$"

$\mathbb{P}^{n-1}_k$: $a_0 = 0$ — points at infinity

Plan for turning this into a scheme:

cover $\mathbb{P}^n$ with $(n+1)$ copies of $\mathbb{A}^n$, corresponding to $\left\{ \underline{[a_0 : \cdots a_n] \in \mathbb{P}^n \mid a_i \neq 0} \right\}$ for $i = 0, \ldots, n$

can scale to make $a_i = 1$.

Call the ith copy of $\mathbb{A}^n_k$

$$X_i = \operatorname{Spec} k[\{x_{j,i}\}_{\substack{0\le j\le n \\ j\neq i}}] \cong \mathbb{A}^n_k$$

Gluing isomorphisms:

$$\begin{array}{ccc}
X_i & & X_j \\
\cup & & \cup \\
D(x_{j,i}) & & D(x_{i,j}) \\
\wr\| & & \wr\| \\
\operatorname{Spec} k[\{x_{k,i}\}][\frac{1}{x_{j,i}}] & \cong & \operatorname{Spec} k[\{x_{k,j}\}][\frac{1}{x_{i,j}}]
\end{array}$$

$$x_{k,i} \longmapsto \frac{x_{k,j}}{x_{i,j}}$$

$$\underbrace{x_{j,i} \longmapsto \frac{1}{x_{i,j}}}$$

informally, "$x_{i,j} = \frac{y_i}{y_j}$"

(check triple intersection compatibility)

These isomorphisms come from wanting

$$[x_{0,i} : x_{1,i} : \cdots : \underbrace{1}_{i} : \cdots : x_{n,i}]$$

$\wr$ equiv in $\mathbb{P}^n$ — scale by $\frac{1}{x_{j,i}} = x_{i,j}$

$$[x_{0,j} : x_{1,j} : \cdots : \underbrace{1}_{j} : \cdots : x_{n,j}] \sim [y_0 : \cdots : y_n]$$

Notes on $\mathbb{P}^n_k$:

For now we can work with $\mathbb{P}^n_k$ via this given affine cover

- Later we'll discuss more general approaches to $\mathbb{P}^n_k$ ((last section in Chapter 4 of Vakil, on $\operatorname{Proj} S_\bullet$) (delaying for a week or two)

- some questions we could already start to consider about $\mathbb{P}^n$, e.g. what the open/closed subsets look like, which opens are affine, etc

Standard computation: $\mathcal{O}_{\mathbb{P}^n_k}(\mathbb{P}^n_k) = k$, so $\mathbb{P}^n_k$ is not affine.

- In fact, this construction defines a scheme $\mathbb{P}^n_A$ for any ring A: interesting to try to think about this from classical perspective

($\mathbb{P}^1_{\mathbb{Z}}$ can be covered by 2 copies of $\operatorname{Spec} \mathbb{Z}[t]$)..

Morphisms of schemes (Ch. 6 in Vakil)

Want to define $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$

We definitely want an underlying cont. map
$$\pi : X \to Y.$$

How do we compare two sheaves on different spaces?
pushforward: $\pi_* \mathcal{O}_X$ and $\mathcal{O}_Y$ are both sheaves on Y, can ask for a sheaf morphism between them.

<u>Two issues:</u>

1) Which direction should the sheaf morphism go?
(not an issue for isomorphisms)

<u>Intuition</u>: $\mathcal{O}_Y$ is "functions on Y", should have some sort of precomposition with $\pi : X \to Y$

So we want a "precomposition rule" or "pullback morphism"
$$\pi^\# : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$$

Explicitly: $\pi^\#(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))$

$(\pi, \pi^\#)$ is a <u>morphism of ringed spaces</u>

2) It turns out that we need a slightly stronger def.

Recall: Schemes are locally ringed spaces, so there is a notion of a section $f \in \mathcal{O}_Y(U)$ vanishing at a point $q \in U$.

Expectation: precomposition ($\pi^\#: \mathcal{O}_Y \to \pi_* \mathcal{O}_X$) should preserve vanishing, in the sense that if f vanished at $q = \pi(p)$, then the pullback $\pi^\#(f)$ should vanish at p.

This expectation is equivalent to requiring that the induced maps on stalks (by $\pi^\#$)

$$\mathcal{O}_{Y,\pi(p)} \longrightarrow \underbrace{(\pi_* \mathcal{O}_X)_{\pi(p)}}_{\text{limit over small open nhoods of } \pi(p)} \longrightarrow \mathcal{O}_{X,p}$$

$\xrightarrow{\pi^{-1}}$ germ at p.

are morphisms of local rings, i.e. $m_{\pi(p)}$ is mapped inside m_p.

(needed for an induced map on residue fields $k_{\pi(p)} \hookrightarrow k_p$).

A morphism of *locally ringed spaces* is one with this additional property.

<u>Def</u>: Morphisms of schemes are morphisms as locally ringed spaces (cont. map, pullback morphism of sheaves satisfying stalk condition)

[Schemes now form a category Sch)

<u>Def</u>: Suppose $\pi: A \to B$ is a ring homomorphism. We've previously defined a cont. map

$$\pi^*: \mathrm{Spec}\, B \to \mathrm{Spec}\, A.$$

Now we promote this to a scheme morphism by defining $\pi^\#: \mathcal{O}_{\mathrm{Spec}\, A} \to (\pi^*)_* \mathcal{O}_{\mathrm{Spec}\, B}$ on the distinguished base: for $f \in A$, take

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Spec}\, A}(D(f)) & & \mathcal{O}_{\mathrm{Spec}\, B}((\pi^*)^{-1}(D(f))) \\ \wr\Vert & & \Vert \\ & & \mathcal{O}_{\mathrm{Spec}\, B}(D(\pi(f))) \\ A\left[\frac{1}{f}\right] & & \wr\Vert \\ & \xrightarrow{\pi} & B\left[\frac{1}{\pi(f)}\right] \end{array}$$

which is well-defined ($D(f) = D(f')$ gives same map) and commutes with restriction.

(Note: This turns Spec into a (contravariant) functor Rings $\to$ Sch)

<u>Prop</u>: Scheme morphisms $\operatorname{Spec} B \to \operatorname{Spec} A$ are precisely those induced by $A \to B$ as above.

<u>Pf</u>: (similar to pset problem 4 about isoms).

<u>Consequence</u>: Any scheme morphism $\pi: X \to Y$ can be formed by gluing together morphisms induced by ring homomorphisms.

Why? We can take an affine open cover $\{\operatorname{Spec} A_i\}$ of Y and then an affine open cover $\{\operatorname{Spec} B_{ij}\}$ of $\pi^{-1}(\operatorname{Spec} A_i)$ for each i, and then π is given by gluing together restrictions

$$\operatorname{Spec} B_{ij} \to \operatorname{Spec} A_i,$$

which must come from ring homomorphisms

$$A_i \to B_{ij}.$$

Examples of morphisms:

1) Let $X = \mathbb{P}^n_k$ and $Y = \operatorname{Spec} k$.

Then there is a natural morphism $X \longrightarrow Y$ given by gluing together morphisms

$$\mathbb{A}^n_k \longrightarrow \operatorname{Spec} k \quad \text{corresponding to}$$

$$k \hookrightarrow k[\{x_{j,i}\}].$$

"X is naturally defined over k"

2) There is a morphism

$$\mathbb{A}^{n+1}_k - \{0\} \longrightarrow \mathbb{P}^n_k$$

"quotient by k^* acting on $\mathbb{A}^{n+1}_k - \{0\}$"

$$p \longmapsto \text{line through } p \text{ (and } 0\text{)}$$