Last time: construction of $\mathcal{O}_{\operatorname{Spec} A}$,

stalks $\mathcal{O}_{\operatorname{Spec} A, [\mathfrak{p}]} \cong A_{\mathfrak{p}}$.

<u>Def</u>: A <u>scheme</u> is a ringed space $X = (X, \mathcal{O}_X)$ such that X can be covered by opens $U_i \subseteq X$ s.t. $(U_i, \mathcal{O}_X|_{U_i}) \cong (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$ for some rings A_i.

(notion of isom of ringed spaces:

homeomorphism $\pi : U_i \xrightarrow{\cong} \operatorname{Spec} A_i$

+ sheaf isomorphism $\pi_*(\mathcal{O}_X|_{U_i}) \cong \mathcal{O}_{\operatorname{Spec} A_i}$.)

Def: An isom of schemes is an isom as ringed spaces.

(morphisms are trickier to define, so we won't have the category Sch of schemes until Thursday)

Defs: An affine scheme is a scheme isom to Spec A for some ring A.

If X is a scheme and $U \subseteq X$ is an open such that $(U, \mathcal{O}_X|_U)$ is affine, then we call U an affine open (or open affine)

Observation: The affine opens form a base for the topology of Spec A (since $D(f) \cong \operatorname{Spec} A[\frac{1}{f}]$ is affine),

and hence for the topology of any scheme.

Consequence: open subsets of schemes are schemes (with the induced ringed space structure $\mathcal{O}_U := \mathcal{O}_X|_U$

"open subscheme"

Def: A *local ring* is a ring A with exactly one maximal ideal.

($\iff$ Spec A has exactly one closed point)

($\iff$ the non-units of A form an ideal)

We sometimes write $A = (A, \mathfrak{m})$ when A is a local ring with max. ideal $\mathfrak{m}$.

Def: The *residue field* of A is $A/\mathfrak{m}$.

Example: $A_{\mathfrak{p}}$ is a local ring for any $\mathfrak{p} \in \operatorname{Spec} A$.

(The maximal ideal is $\mathfrak{p}A_{\mathfrak{p}}$, or denoted $\mathfrak{m}_{\mathfrak{p}}$.)

$(A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} = k_{\mathfrak{p}})$

Def: A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,p}$ is a local ring for all $p \in X$.

Examples: 1) Any scheme is a locally ringed space

$$\mathcal{O}_{\operatorname{Spec} A, [\mathfrak{p}]} \cong A_{\mathfrak{p}}$$

residue fields are all $k_p \cong \mathbb{R}$. [2) continuous maps to $\mathbb{R}$; max. ideals are $\mathfrak{m}_p$ = germs vanishing at p.

In general, if X is a locally ringed space and $f \in \mathcal{O}_X(U)$, and $p \in U$, then we can define the <u>value of f at p</u> is the image of f under

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,p} \longrightarrow \underbrace{k_p := \mathcal{O}_{X,p}/m_p}_{\text{residue field}}$$

In other words, "$f(p) \in k_p$" ← residue field at p.

Compatible with the previous notion of value of $f \in A$ at $[\mathfrak{p}] \in \operatorname{Spec} A$ in the sense that

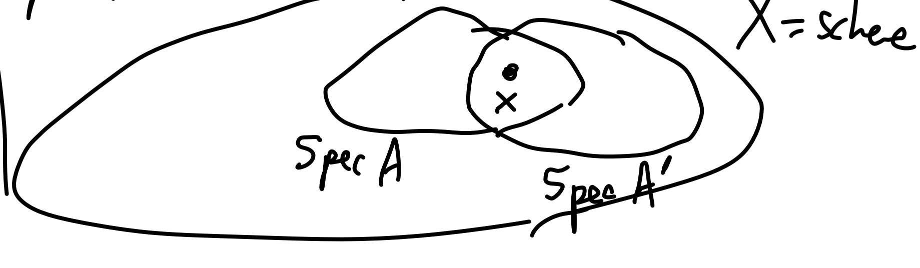
$$A/\mathfrak{p} \xrightarrow{\text{loc.}} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = k_{\mathfrak{p}}$$

$A/\mathfrak{p}$: $\underbrace{\text{prev. notion, domain}}_{\text{noncanonical in the sense of depending on the affine open Spec } A}$

$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$: new location for $f(p)$, field of fractions of

$k_{\mathfrak{p}}$: $\underbrace{\text{residue field at } [\mathfrak{p}]}_{\text{canonical}}$

<u>Lemma</u>: "vanishing loci are closed": $\{p \in U \mid f(p) = 0\}$ is closed.



Examples of non-affine schemes:

$U = D(x) \cup D(y)$

1) "plane minus origin":

$$X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$$

$$U = X - \{[(x,y)]\} \text{ : open subscheme of } X.$$

We saw last time that

$$\mathcal{O}_U(U) := \mathcal{O}_X(U) \cong k[x,y].$$

$\mathcal{O}_X(U) \cong \mathcal{O}_X(X)$

So if $U \cong \operatorname{Spec} A$ for some A, then

$$A = \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \cong \mathcal{O}_U(U) \cong k[x,y]$$

If U was affine, there would exist an isom to $\operatorname{Spec} k[x,y]$ inducing the identity map

$$k[x,y] \longrightarrow k[x,y]$$

But the sections $x, y \in \mathcal{O}_U(U)$ do not simultaneously vanish at any $p \in U$,

while the section $x, y \in \operatorname{Spec} k[x,y]$ do.

Conclusion: U is not affine.

In general: open subschemes of affine schemes might not be affine.

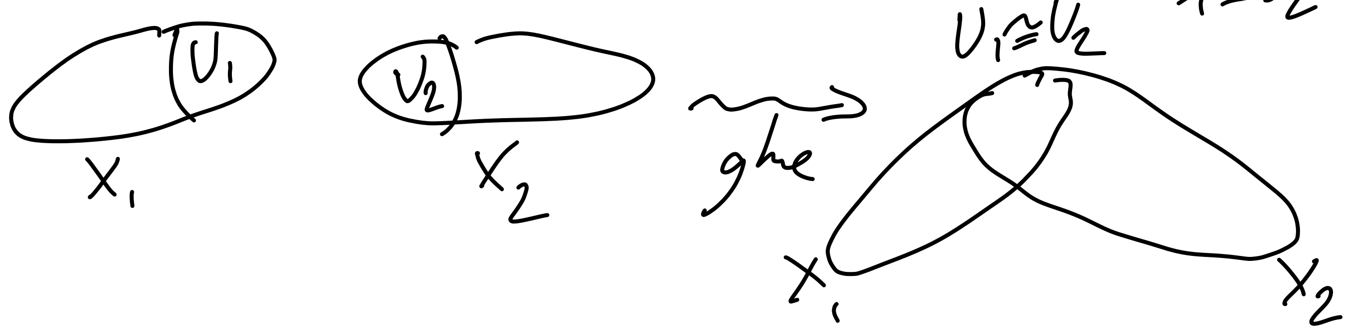
For more examples of schemes, we need to discuss gluing.

Gluing top. spaces: X_1, X_2 top. spaces

S_1, S_2 open subsets of X_1, X_2 respectively

homeomorphism $S_1 \cong S_2$

$\rightsquigarrow$ glue X_1 and X_2 together: $X_1 \sqcup X_2 / \sim_{S_1 \cong S_2}$

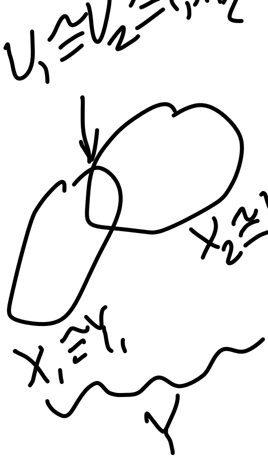


Gluing schemes: similar, but also need to identify $\mathcal{O}_{U_1}$ with $\mathcal{O}_{U_2}$.

"glue sheaves"

Lemma: Let X_1, X_2 be schemes with open subschemes $U_i \subseteq X_i$ $(i=1,2)$. Suppose $\varphi: U_1 \to U_2$ is an isom of schemes.

Then there exists a unique (up to isom.) scheme Y along with open subschemes $Y_1, Y_2 \subseteq Y$ and isomorphisms



$$X_1 \cong Y_1, \quad X_2 \cong Y_2, \quad \underbrace{U_1 \cong Y_1 \cap Y_2 \cong U_2}_{\text{composition is } \varphi}$$

compatible with restriction $U_1 \subseteq X_1$, $U_2 \subseteq X_2$

$Y_1 \cap Y_2 \subseteq Y_1$, $Y_1 \cap Y_2 \subseteq Y_2$.

Note: 1) This generalizes naturally to gluing more than two schemes at once, but requires more 'ation and a triple intersection condition

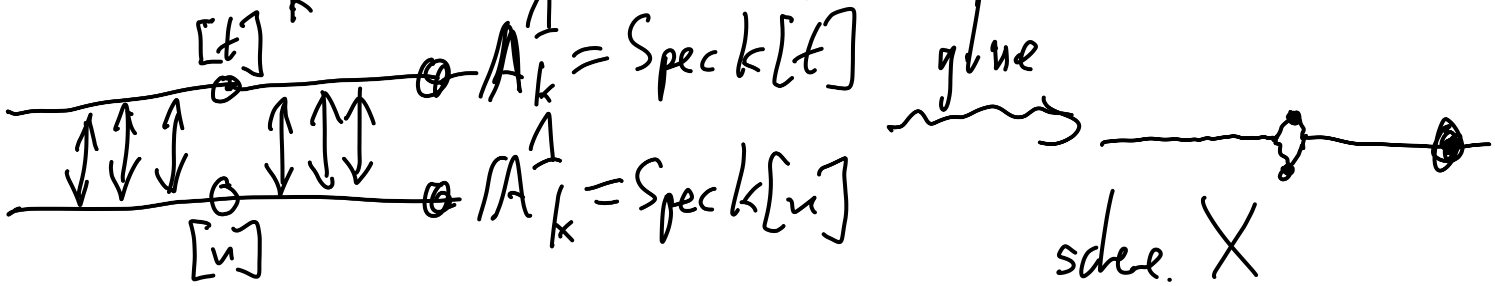
2) This will be the colimit of

$$\begin{tikzcd} & U_1 \arrow[r, hook] \arrow[dl, bend left, "\varphi^{-1}"] & X_1 \\ U_2 \arrow[ur, "\varphi"] \arrow[d, hook] & & \\ X_2 & & \end{tikzcd}$$

Pf: First construct everything in the category Top, then glue sheaves on the glued space using the pset 1 problem. $\blacksquare$

More examples of non-affine schemes:

1) "the line with two origins": glue two copies of $\mathbb{A}^1_k$ via the identity map on $\mathbb{A}^1_k$ - origin:



Can check: $\mathcal{O}_X(X) \cong k[t]$, not affine for similar reason as $\mathbb{A}^2$-origin.

2) "the projective line": same setup, but different isom.

Want to glue $D(t) \subset \operatorname{Spec} k[t]$ to $D(u) \subset \operatorname{Spec} k[u]$

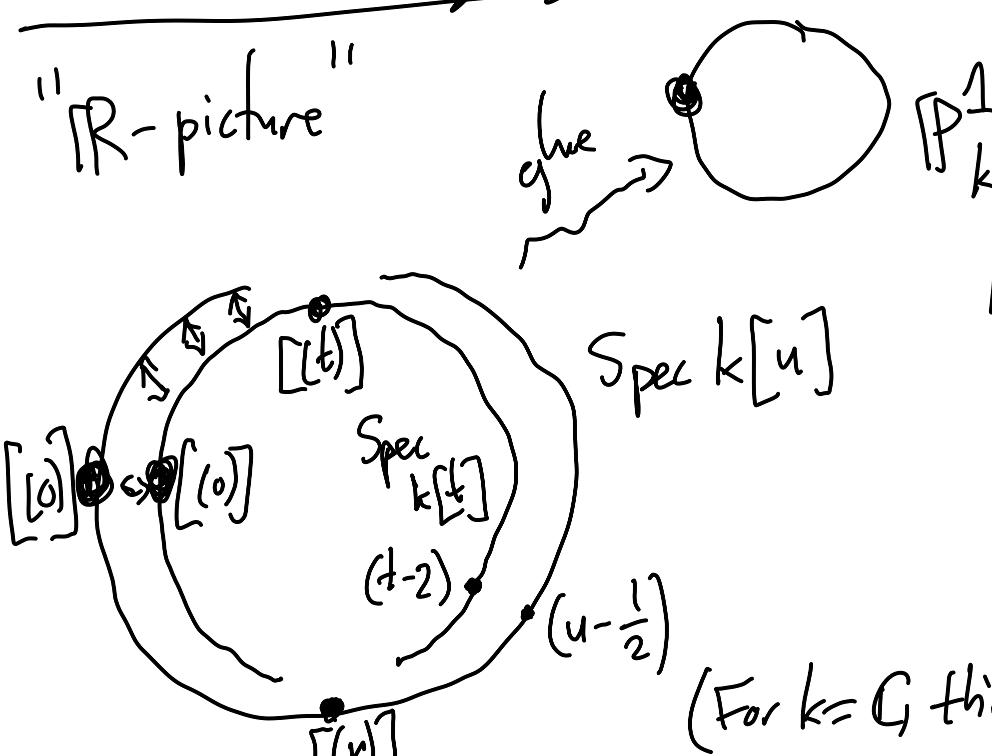
via the isom

$$\begin{array}{ccc} D(t) & & D(u) \\ \wr\!\shortparallel & & \wr\!\shortparallel \\ \operatorname{Spec} k[t, t^{-1}] & \cong & \operatorname{Spec} k[u, u^{-1}] \end{array}$$

isom induced by
$$t \longleftrightarrow u^{-1}$$
$$t^{-1} \longleftrightarrow u.$$

<u>The resulting glued scheme</u> is denoted $\mathbb{P}^1_k$.

"$\mathbb{R}$-picture"



$$[(t-a)] \longleftrightarrow \left[\left(\tfrac{1}{u} - a\right)\right]$$
$$= \left[\left(u - \tfrac{1}{a}\right)\right]$$

$$[(0)] \longleftrightarrow [(0)]$$

(For $k = \mathbb{C}$ this looks like the Riemann sphere)

$$\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) \cong \left\{ (f, g) \in k[t] \times k[u] \;\middle|\; \begin{array}{c} f = g \text{ under} \\ k[t^{\pm 1}] \cong k[u^{\pm 1}] \\ t \longleftrightarrow u^{-1} \end{array} \right\}$$

Only possible if f, g are constant polynomials (and equal),

so $\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) \cong k$.