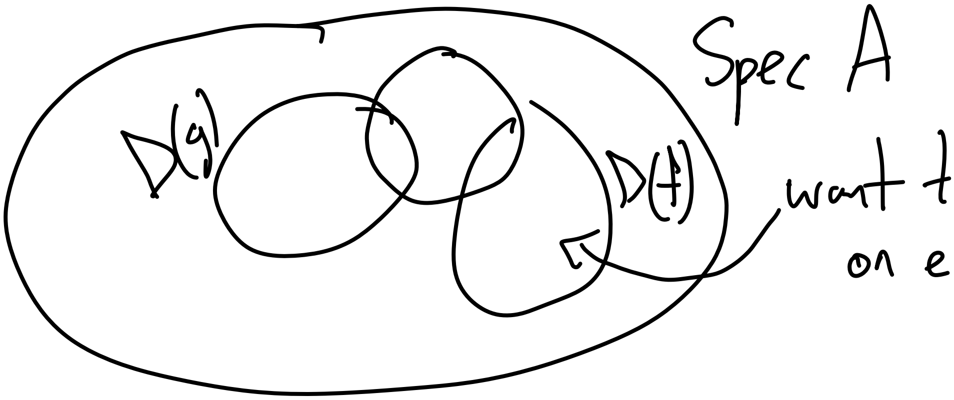
Today: Spec A as a ringed space $(\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$

↑ "structure sheaf"



want to describe sheaf on each distinguished open.

Desired properties:

1) $\mathcal{O}_{\mathrm{Spec}\, A}$ is a sheaf of rings on Spec A.

2) $\mathcal{O}_{\mathrm{Spec}\, A}(\mathrm{Spec}\, A) = A$

3) For any $f \in A$, the homeomorphism

$$D(f) \cong \mathrm{Spec}\, A[\tfrac{1}{f}]$$

extends to a natural isom of ringed spaces

$$(D(f), \mathcal{O}_{\mathrm{Spec}\, A}|_{D(f)}) \underset{\uparrow}{\cong} (\mathrm{Spec}\, A[\tfrac{1}{f}], \mathcal{O}_{\mathrm{Spec}\, A[\frac{1}{f}]}).$$

(Here "natural" can be taken to mean "behaves well under composition", e.g.

$$\begin{tikzcd} D_{\mathrm{Spec}\, A}(fg) \arrow[r, "\cong"] \arrow[d, phantom, "\wr \leftarrow"] & \mathrm{Spec}\, A[\frac{1}{fg}] \arrow[d, phantom, "\wr\ \text{algebraic}"] \\ D_{\mathrm{Spec}\, A[\frac{1}{f}]}(g) \arrow[r, "\cong"] & \mathrm{Spec}\, A[\frac{1}{f}][\frac{1}{g}] \end{tikzcd}$$

)

**Thm:** There is exactly one choice (up to unique isoms)
**Def:** of sheaves $\mathcal{O}_{\operatorname{Spec} A}$ (simultaneously for all $A$) with the above properties.

Pf: Combining properties 2 and 3, we equivalently have isomorphisms of rings

$$\longrightarrow \quad \mathcal{O}_{\operatorname{Spec} A}(D(f)) \cong A\left[\tfrac{1}{f}\right] \text{ for all } A \text{ and } f,$$

along with restriction maps making the diagrams

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} A}(D(f)) \arrow[r, "\text{restrict.}"'] \arrow[d, "\cong"'] & \mathcal{O}_{\operatorname{Spec} A}(D(fg)) \arrow[d, "\cong"'] \\
A\left[\tfrac{1}{f}\right] \arrow[r, "\substack{\text{Loc. inverting} \\ \text{the image of } g \\ \text{in } A[\frac{1}{f}]}"'] & A\left[\tfrac{1}{fg}\right]
\end{tikzcd}$$

commute.

[In particular, taking $f = 1$ gives that restriction map
$\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \longrightarrow \mathcal{O}_{\operatorname{Spec} A}(D(g))$
Looks like localization $A \longrightarrow A\left[\tfrac{1}{g}\right]$

Since the $D(f)$ form a base for the topology of Spec $A$, we can already see that there is at most one such sheaf $\mathcal{O}_{\operatorname{Spec} A}$, since sheaf axioms let us

[ recover $\mathcal{F}\left(\bigcup_{i\in I} U_i\right)$ from $\left\{\mathcal{F}(U_i),\ \mathcal{F}(U_i)\to\mathcal{F}(U_i\cap U_j)\right\}$

$$\mathcal{F}\left(\bigcup_{i\in I} U_i\right) = \left\{ (s_i \in \mathcal{F}(U_i)) \mid s_i \text{ and } s_j \text{ agree on } U_i\cap U_j \right\}$$

$[(s_i)$ agreeing on $U_i\cap U_j]$

(Note: $D(f)\subseteq D(g) \Longrightarrow D(f)=D(fg)$).

so all restriction maps are given.

problem 4 on pset

It remains to check existence of $\mathcal{O}_{\operatorname{Spec} A}$.

Three things to check:

1) <u>compatibility of given data</u>:

If $D(f)=D(g)$, then $A[\frac{1}{f}]$ and $A[\frac{1}{g}]$ are naturally isomorphic: this follows from since $f$ is invertible in $A[\frac{1}{g}]$ and $g$ is invertible in $A[\frac{1}{f}]$, so

$$A\left[\tfrac{1}{f}\right] \cong A\left[\tfrac{1}{f}\right]\left[\tfrac{1}{g}\right] \cong A\left[\tfrac{1}{g}\right]\left[\tfrac{1}{f}\right] \cong A\left[\tfrac{1}{g}\right].$$

This means that we really have (up to canonical isom) a single given value for each $\mathcal{O}_{\operatorname{Spec} A}(U)$ with $U$ a distinguished open, with compatible restriction maps.

2) "base identity": We need to check the identity axiom for covers of one $D(h)$ by other $D(fh)$. By replacing $A[\frac{1}{h}]$ with $A$, we can reduce to

$$\operatorname{Spec} A = \bigcup_{i \in I} D(f_i).$$

Since all restriction maps are linear (e.g. as maps of $A$-modules), it suffices to check that if $s \in A$ $(= \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A))$ vanishes in $A[\frac{1}{f_i}]$ for all $i \in I$, then $s = 0$.

Now, $s$ vanishes in $A[\frac{1}{f_i}] \iff f_i^{e_i} s = 0$ for some $e_i > 0$.
But $D(f_i^{e_i}) = D(f_i)$ cover $\operatorname{Spec} A$, so
$1$ is an $A$-linear combination of the $f_i^{e_i}$,
and hence $s$ is an $A$-linear combination of the $f_i^{e_i} s = 0$,
so $s = 0$.

3) "base gluability": same setup as base identity, a bit harder but similar (see Vakil). ▨

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Warning: $\mathcal{O}_{\operatorname{Spec} A}(U)$ for a general open set $U$ can really be computed only using the sheaf axioms and $\underbrace{\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A[\tfrac{1}{f}]}_{\rightsquigarrow \mathcal{O}_{\operatorname{Spec} A}(U)}$; don't expect a direct formula.

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<u>Note</u>: Given an $A$-module $M$, an analogous result constructs a sheaf $\widetilde{M}$ on $\operatorname{Spec} A$ with $\widetilde{M}(D(f)) = M[\tfrac{1}{f}] := M \otimes_A A[\tfrac{1}{f}]$. (so $\mathcal{O}_{\operatorname{Spec} A} = \widetilde{A}$.)

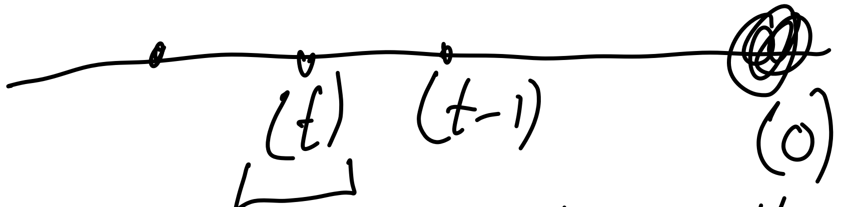
This is an "$\mathcal{O}_{\operatorname{Spec} A}$-module" in the sense that $\widetilde{M}(U)$ is an $\mathcal{O}_{\operatorname{Spec} A}(U)$-module.

## Examples:

0) We can now distinguish between Spec A and Spec B even when they are just single points

Spec k is now not just a point, but a point the constant sheaf k attached.

1) $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$



The open sets in X are all distinguished opens $D(f)$, $f \in k[t]$, so

$$\mathcal{O}_X(D(f)) = k[t]\left[\tfrac{1}{f}\right], \text{ e.g.}$$

$$\mathcal{O}_X(X - \{(t)\}) = k[t, t^{-1}].$$

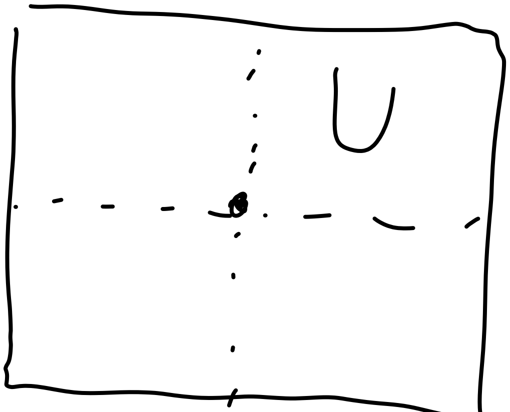
"rational functions are sections of $\mathcal{O}_X$ on nonempty opens"

2) $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$:

$U := \mathbb{A}^2_k - \{[(x,y)]\}$ (complement of origin)

What is $\mathcal{O}_X(U)$?

$U = D(x) \cup D(y)$ (since $(x,y)$ is maximal, so it is the only prime ideal in $k[x,y]$ containing both $x$ and $y$.



As usual, $D(xy) = D(x) \cap D(y)$.

Sheaf axioms:

$$\mathcal{O}_X(U) = \left\{ (f,g) \;\middle|\; \begin{array}{l} f \in \mathcal{O}_X(D(x)),\ g \in \mathcal{O}_X(D(y)), \\ f|_{D(xy)} = g|_{D(xy)} \end{array} \right\}$$

$$= \left\{ (f,g) \in k[x,y,x^{-1}] \times k[x,y,y^{-1}] \;\middle|\; f = g \text{ in } k[x,y,x^{-1},y^{-1}] \right\} = k[x,y]$$

So in this case the restriction map

$$\mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U) \text{ is an isom.}$$

"every rational function defined on $U$ extends to $X$"

(Hartog's theorem).

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<u>Lemma</u>: Let $[\mathfrak{p}] \in \operatorname{Spec} A$. The stalk $\mathcal{O}_{\operatorname{Spec} A, [\mathfrak{p}]}$ is naturally isomorphic to the localization $A_{\mathfrak{p}}$ $\left(= (A \setminus \mathfrak{p})^{-1} A\right)$.

<u>Pf</u>: We can compute the stalk on the distinguished base:

$$\mathcal{O}_{\operatorname{Spec} A, [\mathfrak{p}]} = \left\{ (D(f), s) \;\middle|\; \begin{matrix} s \in \mathcal{O}_{\operatorname{Spec} A}(D(f)) \\ [\mathfrak{p}] \in D(f) \end{matrix} \right\} \Big/ \sim$$

$$= \left\{ (f, s) \;\middle|\; \begin{matrix} f \in A, \; s \in \underbrace{A\left[\tfrac{1}{f}\right]} \\ f \notin \mathfrak{p} \end{matrix} \right\} \Big/ \sim$$

$$= A_{\mathfrak{p}}.$$

▨