Today: a variety of topics on Spec A as top. space

Last time:

- closed subsets $V(I) \subseteq \operatorname{Spec} A$ look like $\operatorname{Spec} A/I$
- distinguished open subsets $D(f) \subseteq \operatorname{Spec} A$ look like $\operatorname{Spec} A[\frac{1}{f}]$

Why? inclusion-preserving bijections

$$\{ \mathfrak{p} \underset{\text{prime}}{\subset} A/I \} \overset{\sim}{\longleftrightarrow} \{ \mathfrak{p} \underset{\text{prime}}{\subset} A \mid I \subseteq \mathfrak{p} \}$$

$$\{ \mathfrak{p} \underset{\text{prime}}{\subset} S^{-1}A \} \overset{\sim}{\longleftrightarrow} \{ \mathfrak{p} \underset{\text{prime}}{\subset} A \mid S \cap \mathfrak{p} = \emptyset \}$$

Here $S^{-1}A$ is the general localization of A at a mult. closed subset $S \subseteq A$:

case $S = \{1, f, f^2, \dots\}$: $S^{-1}A =: A[\frac{1}{f}]$ $(= A_f)$

case $S = A \setminus \mathfrak{p}$, $\mathfrak{p}$ prime. : $S^{-1}A =: A_{\mathfrak{p}}$

(So $\operatorname{Spec} A_{\mathfrak{p}} \overset{\sim}{\longleftrightarrow} \{ \mathfrak{q} \underset{\text{prime}}{\subset} A \mid \mathfrak{q} \subseteq \mathfrak{p} \}$)

Basic topological properties that Spec A might have:

(quasi)compactness: (a top. space is quasicompact if every open cover of it has a finite subcover.

Many people just call this compact, but some people think compact = quasicompact + Hausdorff.

For me: compact = quasicompact)

Lemma: Spec A is quasicompact.

Pf: Since the $D(f)$ are a base, suffices to check distinguished open covers. Then

$$\operatorname{Spec} A = \bigcup_{i \in I} D(f_i) \iff 1 \in \underbrace{(f_i \mid i \in I)}_{\text{ideal gen. by those in } A}$$

$$\iff 1 \in (f_i \mid i \in J) \text{ for some finite } J \subseteq I$$

$$\iff \operatorname{Spec} A = \bigcup_{i \in J} D(f_i). \qquad \blacksquare$$

(Note: $\mathbb{C}$ is not compact with the usual topology, but $\operatorname{Spec} \mathbb{C}[t] =: \mathbb{A}^1_{\mathbb{C}}$ (affine line over $\mathbb{C}$) is compact.)

Connectedness:

Easy to check that

$$\operatorname{Spec}(A_1 \times A_2) \cong \operatorname{Spec} A_1 \sqcup \operatorname{Spec} A_2$$

and hence is disconnected (for $A_1, A_2 \neq 0$).

Converse ($\operatorname{Spec} A$ disconnected $\Rightarrow A \cong A_1 \times A_2$) is true but harder:

- algebra exercise
- immediate once we have constructed the promised sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$

Irreducibility:

Def: X is reducible if $X = X_1 \cup X_2$ for proper closed subsets $X_1, X_2 \subset X$.

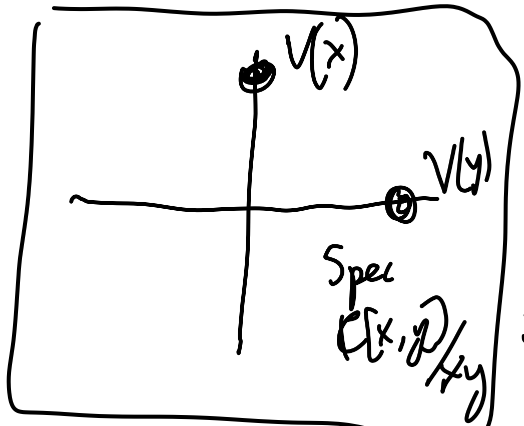
(irred $\Rightarrow$ connected).

Example: $\mathbb{C}$ with usual topology is connected but reducible.

(equiv. def of reducibility: there are disjoint nonempty opens)

Example: If A is a domain, then $\operatorname{Spec} A$ is irreducible because every open contains the dense point $[(0)]$.

Example: $\mathrm{Spec}\left(\mathbb{C}[x,y]/xy\right)$ is connected but reducible.



$= V(xy)$

$\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[x,y]$

reducible because

$$V(xy) = \underbrace{V(x)}_{} \cup \underbrace{V(y)}_{}, $$

2 irred components.

Def: An irred. component of X is a maximal irred. subset of X.

Noetherianity:

Def: A ring A is Noetherian if its ideals satisfy the ascending chain condition (ACC):

$$I_1 \subseteq I_2 \subseteq \dots \subseteq A \implies I_i = I_{i+1} \text{ for } i \gg 0.$$

Note: There are lots of Noetherian rings:

- $\mathbb{Z}$ is Noetherian
- fields are Noetherian
- A Noetherian $\implies A/I,\ S^{-1}A$ are Noetherian.
- A Noetherian $\implies A[t]$ is Noetherian "Hilbert basis theorem"

Def: A top. space X is Noetherian if its closed subsets satisfy the DCC:

$$X \supseteq \underset{\text{closed}}{Z_1} \supseteq \underset{\text{closed}}{Z_2} \supseteq \cdots \Longrightarrow Z_i = Z_{i+1} \text{ for } i \gg 0.$$

Can check: A Noetherian $\Longrightarrow$ Spec A is Noetherian.

Thm: Suppose X is a Noetherian top. space. Then each closed subset $Z \subseteq X$ can be uniquely (up to reordering) written in the form

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_n,$$ where the Z_i are irred closed subsets and $Z_i \not\subseteq Z_j$ for $i \neq j$.

Pf: Exercise with Noetherianity condition.

Summary:

$$X = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n] =: \mathbb{A}^n_{\mathbb{C}}$$ "affine n-space" over $\mathbb{C}$

if replaced by a general Noetherian ring A

- X is quasicompact, irreducible, Noetherian

have finitely many irred. components

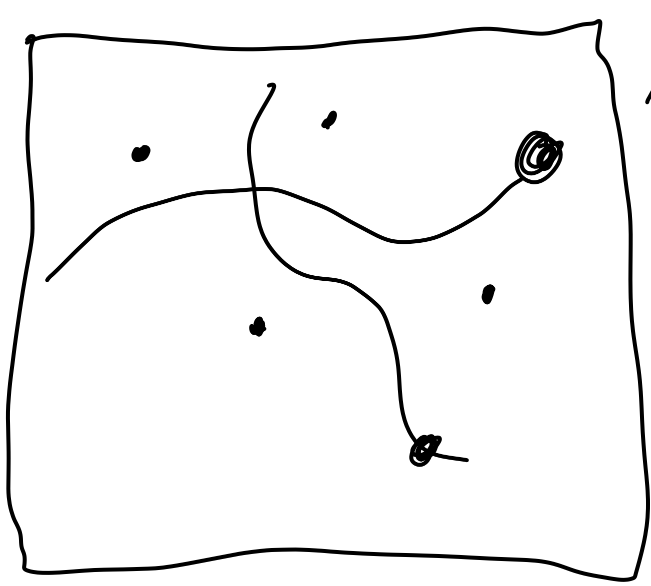
- closed points of $X \longleftrightarrow$ maximal ideals in $\mathbb{C}[x_1, \dots, x_n]$

$\longleftrightarrow (x_1 - a_1, \dots, x_n - a_n)$ for $(a_1, \dots, a_n) \in \mathbb{C}^n$

$\underbrace{\text{Zariski's lemma}}$ this says that if $\mathfrak{m} \subset \mathbb{C}[x_1, \dots, x_n]$ is maximal, then $\mathbb{C}[x_1, \dots, x_n]/\mathfrak{m} \cong \mathbb{C}$.

needs alteration

- closed subsets are finite unions of irred. closeds (by Noetherianity)
  - with a little more work can show that $Z \subset X$ is irred. closed $\Longleftrightarrow Z = \overline{\{p\}}$ for some $p \in X$.



$\mathbb{A}^2_{\mathbb{C}}$

$\cup$ closed

Z

"looks like a finite union of points, curves, surfaces, etc."

Correspondences between ideals in A and subsets of $\operatorname{Spec} A$.

We have:

$$V : \{\text{ideals in } A\} \longrightarrow \{\text{closed sets in } \operatorname{Spec} A\}$$

$$I \longmapsto V(I) = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq I\}$$

We can define a function in the reverse direction:

$$I : \{\text{subsets in } \operatorname{Spec} A\} \longrightarrow \{\text{ideals in } A\}$$

$$S \longmapsto I(S) := \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$$

"functions vanishing on X"

Thm: (i) For $S \subseteq \operatorname{Spec} A$, $V(I(S)) = \overline{S}$ (top. closure)

(ii) For $J \underset{\text{ideal}}{\subseteq} A$, $I(V(J)) = \sqrt{J}$

$$\sqrt{J} = \{a \in A \mid a^d \in J \text{ for some } d > 0\}$$

$J = \sqrt{J}$

(iii) V and I define a bijection between radical ideals in A and closed subsets in $\operatorname{Spec} A$.

Part (ii) of the theorem is just a restatement of the algebra result

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \supseteq I \\ \text{prime}}} \mathfrak{p}.$$

$$\left(\text{this follows from } \sqrt{0} = \bigcap_{\substack{\mathfrak{p} \subset A \\ \text{prime}}} \mathfrak{p} \right)$$

nilradical = {nilpotent elts}

ring theory	scheme theory
ring A	affine scheme $\operatorname{Spec} A$
radical ideal $I = \sqrt{I}$ $\xleftrightarrow{\text{Thm (iii)}}$	closed subset $V(I)$
prime ideal $\mathfrak{p}$	irred closed subset $\overline{\{[\mathfrak{p}]\}}$
maximal ideal $\mathfrak{m}$	closed point $[\mathfrak{m}]$
localization $A[\frac{1}{f}]$	distinguished open $D(f)$
coming later: element $f \in A$	global section $f \in \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$, with $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$
ideal I	closed subscheme $V(I) \cong \operatorname{Spec}(A/I)$
localization $A_{\mathfrak{p}}$ ($\mathfrak{p} \subset A$ prime)	stalk $(\mathcal{O}_{\operatorname{Spec} A})_{[\mathfrak{p}]}$
"rings glued together"	scheme formed by gluing affine schemes
A-module M	quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} A$