Last time: $\operatorname{Spec} A := \{\text{prime ideals } \mathfrak{p} \subset A\}$, some examples/pictures

Today: topology on Spec A

Recall that the plan is to eventually construct a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ on Spec A. One feature of this construction will be that

$$\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \qquad \text{(global sections)}$$

is precisely A.

In other words, Spec A should be a construction of a space such that some "ring of functions" on the space is A.

Language/Defs: Suppose $f \in A$. Then f is a "function on Spec A"; if $\mathfrak{p} \in \operatorname{Spec} A$, then the "value of f on $\mathfrak{p}$" is "$f \bmod \mathfrak{p}$", i.e. the image of f under $A \to \underbrace{A/\mathfrak{p}}$.

Example: $A = \mathbb{C}[t]$, $f \in \mathbb{C}[t]$ is a polynomial

Spec $A =$ 

Then the value of f at $(t-a)$ is the image of f under $\mathbb{C}[t] \to \mathbb{C}[t]/(t-a) \cong \mathbb{C}$

which is just $f(a) \in \mathbb{C}$

The value of f at (0) is just f itself.

Def (of the topology on Spec A)

Let A be a ring. Then a subset of Spec A is **closed** if and only if it is equal to a "vanishing locus"

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec} A \mid S \subseteq \mathfrak{p}\}$$

for some subset $S \subseteq A$.

("Locus where all elements of S vanish")

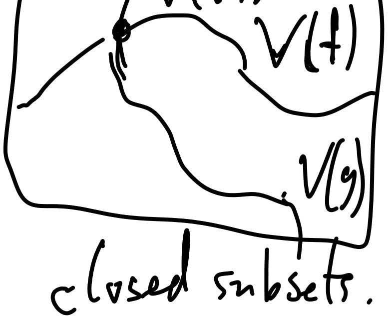
Lemma: This defines a topology on Spec A

Pf: $\emptyset = V(1)$, $\operatorname{Spec} A = V(\emptyset)$ are closed.

$$\bigcap_{i \in I} V(S_i) = V\Big(\bigcup_{i \in I} S_i\Big)$$

$\longrightarrow$ $V(f) \cup V(g) = V(fg)$ (by def of prime ideal)

(**Notation**: $V(f) = V(\{f\})$) $\blacksquare$



Spec A

closed subsets.

Alternate perspective: Let $D(f) := \operatorname{Spec} A \setminus V(f)$

$= \{ \mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \}$. This is called a distinguished open. "does not vanish locus"

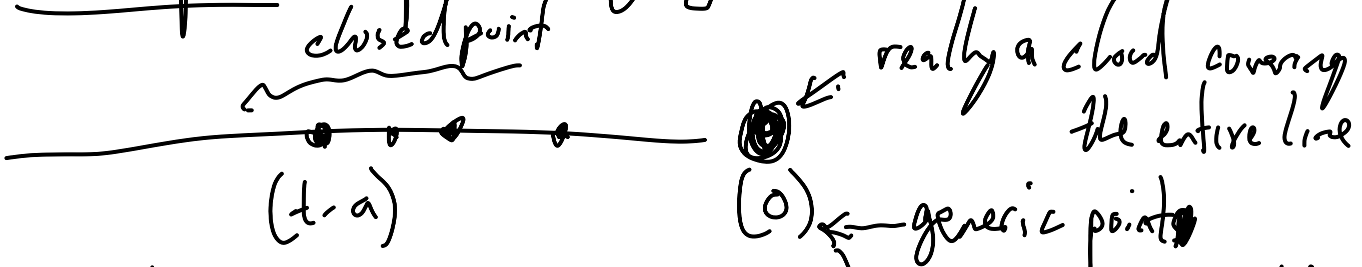
Lemma: The distinguished opens form a base for the topology of Spec A (that is, every open set is a union of distinguished opens)

Pf: $\operatorname{Spec} A \setminus V(S) = \bigcup_{f \in S} D(f)$. $\blacksquare$.

Note: This is an especially well-behaved base in the sense that it is closed under finite intersection:

$D(f) \cap D(g) = D(fg)$.

Example: Spec $\mathbb{C}[t]$:



closed sets: $V((t-a_1)\cdots(t-a_d)) = \{(t-a_1),\ldots,(t-a_d)\}$,

so proper closed sets of Spec $\mathbb{C}[t]$ are the finite sets of Spec $\mathbb{C}[t] \setminus \{(0)\}$.

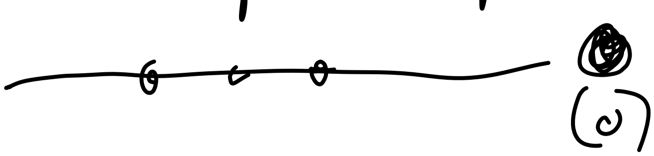
Close to the cofinite topology, but every nonempty open set contains the point (0).

In other words, (0) is dense in Spec $\mathbb{C}[t]$, so this point is arbitrarily close to every point on the line.

(Note $(0) \in D(f(t))$ for every nonzero polynomial f.)

Easy generalization: if A is a domain, then (0) is dense in Spec A.

Other examples: Spec $\mathbb{Z}$, Spec $k[t]$ have the same topology (cofinite + dense point).



Notation: we will sometimes write
$[\mathfrak{p}]$ instead of $\mathfrak{p}$ to denote
the point in Spec A corresponding to the
prime ideal $\mathfrak{p} \subset A$.

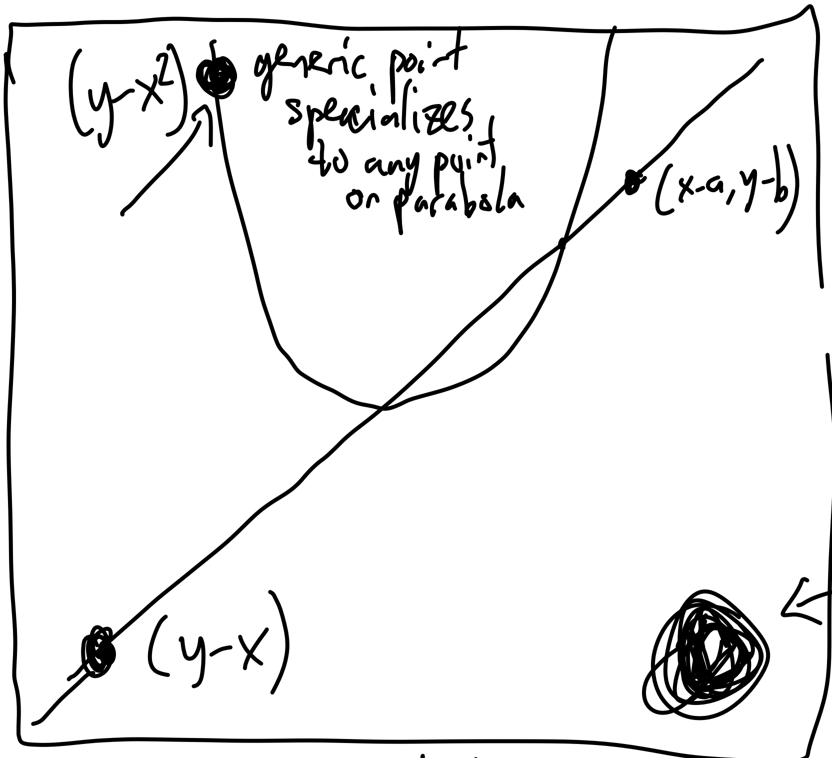
Lemma: Let $\mathfrak{p}, \mathfrak{q} \subset A$ be prime ideals. Then

$$[\mathfrak{q}] \in \overline{\{[\mathfrak{p}]\}} \iff \mathfrak{q} \supseteq \mathfrak{p}.$$

Pf: $\overline{\{[\mathfrak{p}]\}} = V(\mathfrak{p})$. ∎

In this case we say that $[\mathfrak{q}]$ is a <u>specialization</u> of $[\mathfrak{p}]$. If the point $[\mathfrak{p}]$ is not closed (i.e. $\mathfrak{p}$ is not maximal), we say that it is a <u>generic point</u>.

Example: $\operatorname{Spec} \mathbb{C}[x,y] =: \mathbb{A}^2_{\mathbb{C}}$ = "affine plane over $\mathbb{C}$"



Can check:
$[(x-a, y-b)]$ is in the closure of $[(y-x^2)]$
$\iff b-a^2=0$.

(0), generic point dense in $\operatorname{Spec} \mathbb{C}[x,y]$

"small open nhoods" here are $D(f)$ for $f \in \mathbb{C}[x,y]$,
i.e. complements of curves cut out by polynomials

("small" open sets are large and tend to intersect each other)

Induced maps on Spec:

Suppose $\pi: A \to B$ is a ring homomorphism. Then there is a natural function

$$\begin{array}{rcl} \pi^*: \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ \mathfrak{p} & \longmapsto & \pi^{-1}(\mathfrak{p}) \\ \cap & & \cap \\ B & & A \end{array}$$

this makes Spec into a (contravariant) functor Rings $\to$ Sets.

(Why is $\pi^{-1}(\mathfrak{p})$ also prime?)

$A/\pi^{-1}(\mathfrak{p}) \to B/\mathfrak{p}$ is injective, so

$B/\mathfrak{p}$ is a domain $\Rightarrow A/\pi^{-1}(\mathfrak{p})$ is a domain

<u>Lemma</u>: $\pi^*: \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous.

<u>Pf</u>: $(\pi^*)^{-1}(D(f)) = D(\pi(f))$. $\blacksquare$

(So now Spec: Rings $\to$ Top).

Many examples of ring homomorphisms to think about "on Spec"

(e.g. what does $\mathbb{R}[t] \hookrightarrow \mathbb{C}[t]$ look like?)

<u>Lemma</u> ("closed subsets of affine schemes look like affine schemes")

Suppose A is a ring and $I \subseteq A$ is an ideal. Then the map $\operatorname{Spec} A/I \to \operatorname{Spec} A$ induced by

$$A \to A/I$$

has image $V(I)$ and induces a homeomorphism onto the image.

(identification of $V(I)$ with $\operatorname{Spec} A/I$).

<u>Lemma</u> ("distinguished open subsets of affine schemes look like affine schemes")

. . . . .

(identification of $D(f)$ with $\operatorname{Spec} A[\frac{1}{f}]$)

$(A[\frac{1}{f}] = A_f =$ localization of A wrt $\{1, f, f^2, \dots\})$

Problem sets:

- will post pset 2 (first graded one) this afternoon, due a week from today.
- email submissions to me, subject line "631 Problem Set N"
- welcome to discuss problems with others, but should write up solutions independently and acknowledge people you worked with.