(done with sheaf theory for now — this was quick, but want to get ring theory involved)

Today: - motivation/plan
- definition of Spec A (as a set)

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Motivation:

A ringed space is a pair $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a sheaf of rings on $X$ (restriction maps are ring homoms, stalks are rings, etc)

We think of $\mathcal{O}_X(U)$ as some "ring of functions defined on $U$" and encodes some structure we care about on $X$.

Suppose $X \subseteq \mathbb{C}^n$.

- if we just care about the subspace topology of $X$, take $\mathcal{O}_X$ to be cont. functions to $\mathbb{R}$.
- if we know $X$ is a smooth immersed submanifold, could take $\mathcal{O}_X$ to be smooth functions to $\mathbb{R}$.

- Classical alg. geom: if $X \subseteq \mathbb{C}^n$ is defined by polynomial equations, might take

$$\mathcal{O}_X(U) = \left\{ \text{functions } f : U \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ agrees with} \\ \text{some rational} \\ \text{function} \\ \text{in } \underbrace{\mathbb{C}(x_1, \ldots, x_n)} \end{array} \right\}$$

- We will essentially do this, but:

Zariski topology.

- the subspace topology from $\mathbb{C}^n$ is bad here for a few reasons:
  - fails gluability with usual topology:

disjoint opens

$U_1$ $U_2$ . . . .

    - take different poly on each
    - can't glue $\infty$-ly many.

  too many disjoint opens in $\mathbb{C}^n$

  - we don't really care about open/closed subsets of $\mathbb{C}^n$ unless they are defined by polynomials
- want to generalize to fields other than $\mathbb{C}$ or $\mathbb{R}$, so want a natural topology on $k^n$ other than the discrete topology.

- the underlying set of $X \subseteq \mathbb{C}^n$
(cut out by poly. equations)

isn't optimal for us.

add points to ensure good structure.

- we are already using unusual topology, and adjusting the set helps us talk about that topology.
- If $k$ is not alg. closed ($k \neq \bar{k}$), then "$k^n$ should include some features of $\bar{k}^n$"
(to avoid issues where polynomials have very few or no solutions over $k$)

- we don't want to be stuck thinking about subsets $X \subseteq k^n$ (want more general/intrinsic defs to help with gluing etc.)

Plan:

construction:

$$\text{ring } A \rightsquigarrow \left.\begin{array}{l} \text{- set } \operatorname{Spec} A \\ \text{- Zariski topology on } \operatorname{Spec} A \\ \text{- sheaf of rings } \mathcal{O}_{\operatorname{Spec} A} \end{array}\right] \begin{array}{c} \underline{\text{affine}} \\ \text{scheme} \\ \operatorname{Spec} A \end{array}$$

When $A = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, get something like the classical picture above.

Once we have the full construction of
$\operatorname{Spec} A = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, schemes will
just be ringed spaces that can be covered by
affine schemes.

(manifold analogy: scheme $\longleftrightarrow$ manifold
affine scheme $\longleftrightarrow$ small open set in $\mathbb{R}^n$)

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Today: Spec $A$ as a set
Thursday: as a top. space

Def: Let $A$ be a ring (commutative, associative, with $1$). Then $\operatorname{Spec} A$, as a set, is the set of prime ideals of $A$.

(Recall: an $I \subseteq A$ is prime $\Longleftrightarrow$ ($I \neq A$ and $xy \in I \Rightarrow x \in I$ or $y \in I$) $\Longleftrightarrow$ $A/I$ is a domain, i.e. a nonzero ring with no zero-divisors ($xy = 0 \Rightarrow x = 0$ or $y = 0$))

Basic facts about prime ideals:

1) Maximal ideals are prime. ($I \subseteq A$ is maximal if $I$ is maximal among ideals not equal to $A$ itself, $\Longleftrightarrow A/I$ is a field).

2) Any nonzero ring has at least one maximal ideal, and hence at least one prime ideal.
(Pf: Apply Zorn's Lemma to the set of proper ideals)

So $\operatorname{Spec} A = \emptyset \Longleftrightarrow A = 0$.

Examples of Spec:

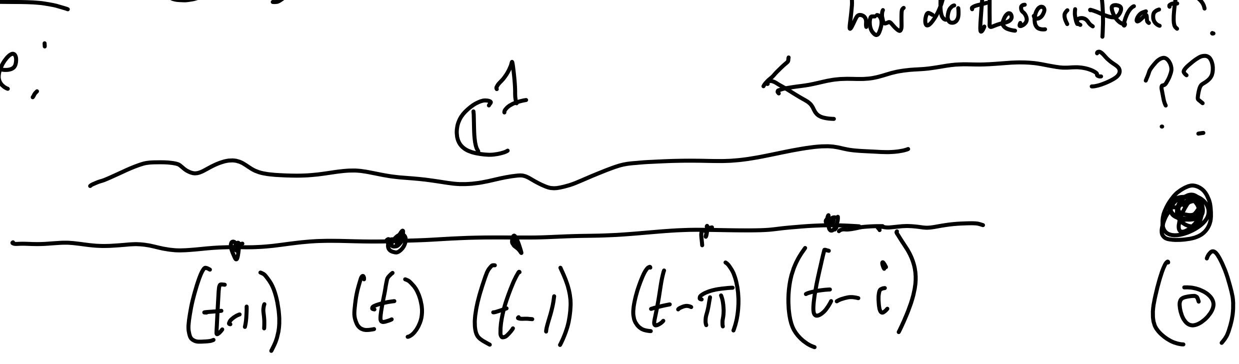
0) $\operatorname{Spec} 0 = \emptyset$

1) If $k$ is a field, then $\operatorname{Spec} k = \{\underset{\overset{\|}{\mathcal{O}}}{(0)}\}$ is just a single point (ideals in $k$ are $(0)$ and $k$),

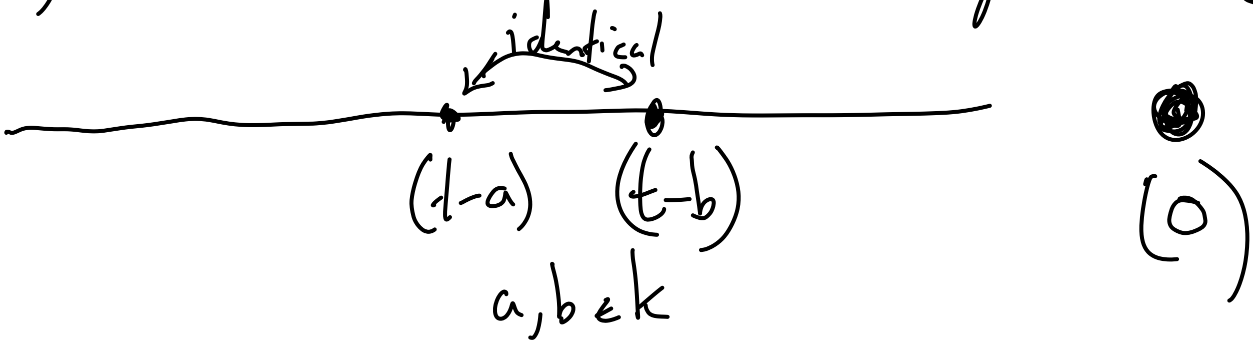
1.5) This is not an if and only if, e.g.
$\operatorname{Spec} k[t]/t^2 = \{(t)\}$ but $k[t]/t^2$ is not a field.

2) $A = \mathbb{C}[t]$: This is a PID, so all ideals are of the form $(f)$ for some $f \in A$. If $\deg f > 1$, then this is not prime because $f$ is reducible (since $\mathbb{C} = \overline{\mathbb{C}}$). So the only prime ideals are:
$$(0),\ (t-a) \text{ for } a \in \mathbb{C} = \mathbb{C}^1$$

picture:



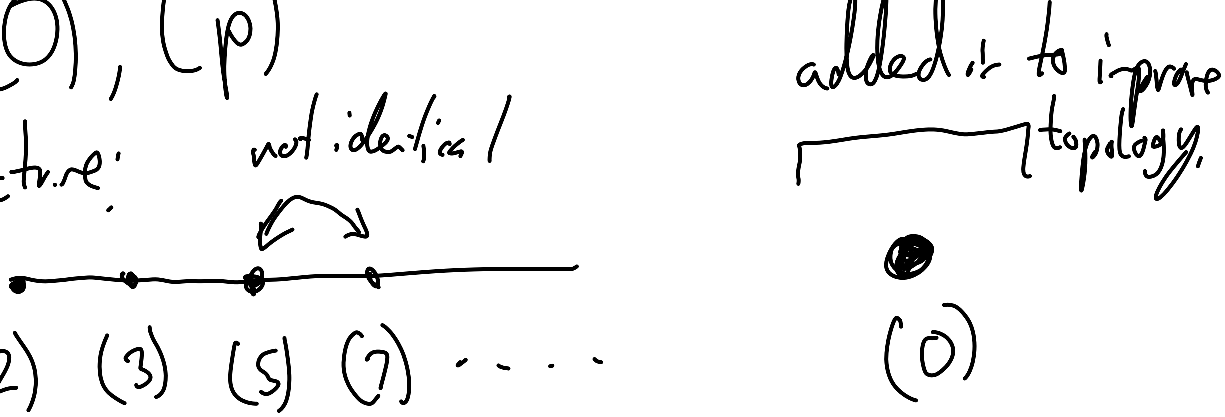
2.1) $A=k[t]$, $k=\bar{k}$: same story as $\mathbb{C}[t]$




3) $A=\mathbb{Z}$: again a PID, prime ideals are

$(0)$, $(p)$

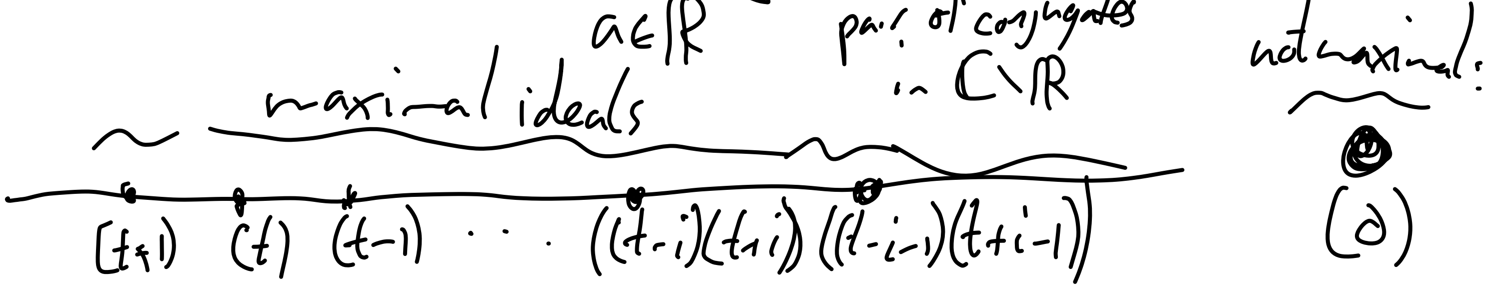
Similar picture:



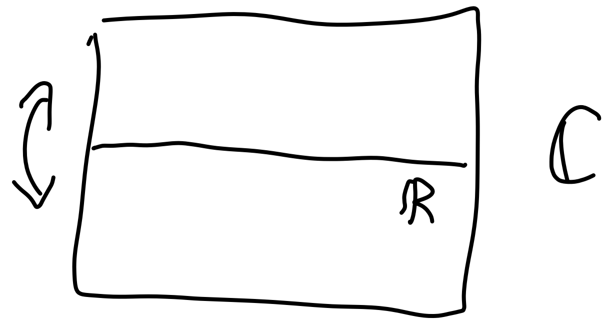

4) $A=\mathbb{R}[t]$: still a PID, but there are now additional prime ideals $(f)$ for $f$ quadratic irred/$\mathbb{R}$

$(f=(t-z)(t-\bar{z}),\ z\in\mathbb{C}\setminus\mathbb{R})$

3 types of points: $\boxed{\underbrace{(t-a)}_{a\in\mathbb{R}}}$, $\underbrace{((t-z)(t-\bar{z}))}_{\text{pair of conjugates in } \mathbb{C}\setminus\mathbb{R}}$, $(0)$



So $\operatorname{Spec}\mathbb{R}[t] = \underbrace{\mathbb{R} \sqcup (\mathbb{C} \setminus \mathbb{R})/_{z\sim\bar{z}}}_{\mathbb{C}/_{z\sim\bar{z}}} \sqcup \{\bullet\}$




"$\underline{\operatorname{Spec}\mathbb{C}[t] \text{ is a}}$ double cover of $\operatorname{Spec}\mathbb{R}[t]$".

5) $A = \mathbb{C}[x,y]$: not a PID, have to work a bit harder to classify prime ideals:

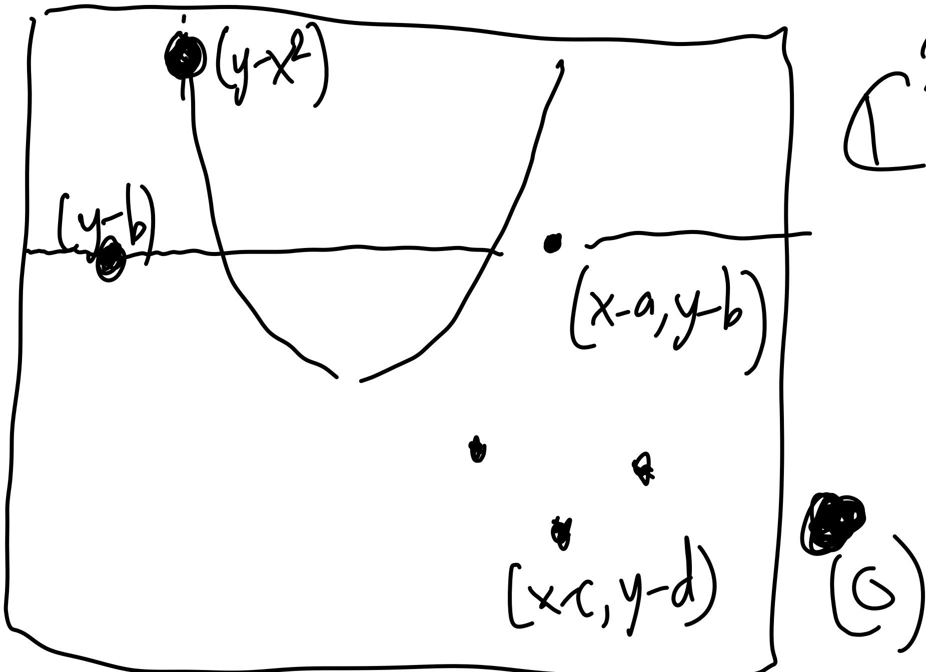
(Can show that if $\mathfrak{p} \subset A$ is prime and contains two elements $f$ and $g$ that are irreducible polynomials that are not multiples of each other, then $\mathfrak{p} = ((x-a),(y-b))$ for some $a,b \in \mathbb{C}$.)

Then there are 3 types of prime ideals in $A$:

1) $\mathfrak{p} = ((x-a),(y-b)) \longleftrightarrow (a,b) \in \mathbb{C}^2$

2) $\mathfrak{p} = ((f))$, $f$ irred $\longleftrightarrow$ curves in $\mathbb{C}^2$ cut out by a single polynomial,

3) $\mathfrak{p} = (0)$.




$\mathbb{C}^2$

**Spec $\mathbb{C}[x,y]$**

- one point for each point in $\mathbb{C}^2$
- one point for each irred. curve in $\mathbb{C}^2$
- zero ideal

$$\left\{\substack{\text{maximal ideals}\\ \text{in } A}\right\} = \{(x-a, y-b) \mid a,b \in \mathbb{C}\} \cong \mathbb{C}^2$$

Just as $(0)$ was an additional point corresponding to the line Spec $\mathbb{C}[t]$,

$(f)$ should be an additional point corresponding to the curve $\{(x-a, y-b) \mid f(a,b)=0\}$.

6) $A = \mathbb{C}[x_1, \ldots, x_n]$ : imagine this as

"$\mathbb{C}^n \sqcup$ additional points"

$\mathbb{C}^n \longleftrightarrow (x_1-a_1, x_2-a_2, \ldots, x_n-a_n)$ maximal ideals

additional points $\longleftrightarrow$ non-maximal prime ideals $\Big\}$ hard to list, but correspond to curves, surfaces, etc inside $\mathbb{C}^n$.

$$\{\text{maximal ideals in } \mathbb{R}[x,y]\} \longleftrightarrow \mathbb{C}^2/(z_1,z_2)\sim(\bar{z}_1,\bar{z}_2)$$

natural bijection

$$(x-a,\ y-b) \longleftrightarrow (a,b)$$

$a,b\in\mathbb{R}$ $\qquad$ $a,b\in\mathbb{R}$

$\mathbb{C}^2 = \{\text{max. ideals in } \mathbb{C}[x,y]\}$

$m\subset\mathbb{R}[x,y]$ maximal

$$\Longleftrightarrow \mathbb{R}[x,y]\to\underbrace{\mathbb{R}[x,y]/m}_{\text{is a field}}$$

$\updownarrow$

surjective ring homomorphisms $\mathbb{R}[x,y]\to k$, $(k=\mathbb{R} \text{ or } \mathbb{C})$

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Can think about $\operatorname{Spec}\mathbb{Z}[x]$