Last time: – categories Sets_X, Rings_X, Ab_X, Ab_X^{pre}
– notion of an abelian category
(can add morphisms, $\underbrace{\oplus, \ker, \mathrm{coker}}_{\text{exist}}$,
additional properties)

Claim: Ab_X^{pre} is an abelian category.

Why? Can do everything on the level of $\mathcal{F}(U)$, e.g.

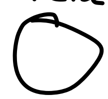
$$(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U) \longleftarrow$$

$$(\ker \pi)(U) = \ker(\pi(U) : \mathcal{F}(U) \to \mathcal{G}(U)).$$

What about Ab_X? Issue is that the sheaf axioms aren't directly compatible with doing things "open set by open set"

Can check: $\oplus$, $\ker$ are still fine (the presheaf construction actually produces a sheaf)

But there's an issue with coker:

<u>Example:</u> X = top. space ($X = S^1 = \mathbb{R}/\mathbb{Z}$ circle )

$\mathcal{M}_Y$ = sheaf on X of cont. functions to Y = top. space.

Note: (1) if $\pi: Y \to Z$ is a cont. map, then it induces a morphism of sheaves $\mathcal{M}_Y \to \mathcal{M}_Z$.

(2) if Y has the structure of an abelian group then $\mathcal{M}_Y$ is a sheaf of abelian groups.

Consider the maps

$$\mathcal{M}_{\mathbb{Z}} \xrightarrow{f} \mathcal{M}_{\mathbb{R}} \xrightarrow{g} \mathcal{M}_{\mathbb{R}/\mathbb{Z}}$$

Claim: 1) the presheaf cokernel of f fails glueability

2) the presheaf cokernel of g fails identity

} sheaf axioms, for $X = S^1$.

2): Let $\mathcal{F}$ = presheaf cokernel of g
$= \operatorname{coker}^{pre} g$

$$\mathcal{F}(U) = \{U \xrightarrow{cont} \mathbb{R}/\mathbb{Z}\} / \left(\text{image of } \{U \to \mathbb{R}\} \to \{U \to \mathbb{R}/\mathbb{Z}\}\right)$$

Get: $\mathcal{F}(U) = \begin{cases} 0 & \text{if } U \subsetneq X = \mathbb{R}/\mathbb{Z} \\ \mathbb{Z} & \text{if } U = X \end{cases}$

Fails identity for $\mathcal{F}(X)$.

But it turns out that any morphism in Ab_X does have a cokernel inside Ab_X; it just isn't necessarily equal to the cokernel inside Ab_X^{pre} (and takes more work to construct)

Def: Let $\mathcal{F}$ be a presheaf on X. A sheafification of $\mathcal{F}$ is a sheaf $\mathcal{F}^{sh}$ on X along with a (presheaf) morphism $sh: \mathcal{F} \to \mathcal{F}^{sh}$ s.t. for any sheaf $\mathcal{G}$ on X and presheaf morphism $\mathcal{F} \xrightarrow{\tau} \mathcal{G}$, τ factors uniquely through sh:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[dr, "\tau"'] & \mathcal{F}^{sh} \arrow[d, dotted, "\exists!"] \\ & \mathcal{G} \end{tikzcd}$$

univ. prop, so unique if it exists.

("adjoint to forgetful functor")

$$\operatorname{Mor}_{Sets_X^{pre}}(\mathcal{F}, \mathcal{G}) \underset{\text{canonically}}{\cong} \operatorname{Mor}_{Sets_X}(\mathcal{F}^{sh}, \mathcal{G})$$

$\tau \in$

Prop: Let $\mathcal{F}, \mathcal{G}$ be sheaves of ab. groups on X and let $\pi: \mathcal{F} \to \mathcal{G}$ be a morphism.

Let $\mathcal{H}$ be the presheaf cokernel $\mathrm{cok}^{pre}\pi$.

Suppose $\mathcal{H}^{sh}$ is a (the) sheafification of $\mathcal{H}$.

Then $\mathcal{H}^{sh}$ is the cokernel of π in Ab_X.

(really have $\mathcal{G} \xrightarrow{\iota} \underset{\mathcal{H}}{\overset{\shortparallel}{\mathrm{cok}^{pre}\pi}}$ and compose with $sh: \mathcal{H} \to \mathcal{H}^{sh}$

to get $\mathcal{G} \xrightarrow{sh \circ \iota} \mathcal{H}^{sh}$, and that should be the cokernel)

Pf: Recall: cokernel(π) is the colimit of

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\pi"] \arrow[d] \arrow[dr] & \mathcal{G} \arrow[d] \\ 0 \arrow[r] & \underbrace{\mathcal{H}}_{\text{colimit}} \end{tikzcd}$$

.

$$\sigma \circ \pi = 0$$

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\pi"] \arrow[d] & \mathcal{G} \arrow[d, "\tau"] \arrow[dr, bend left, "sh \circ \tau"] \arrow[ddr, bend left=60, "\sigma"] & \\
0 \arrow[r] \arrow[dr, bend right] \arrow[ddr, bend right=50] & \mathcal{H} = \operatorname{coker}^{pre} \pi \arrow[r, "sh"] \arrow[dr, dashed, "\exists! \text{ by } \operatorname{coker}^{pre} \text{ univ. prop}"'] & \mathcal{H}^{sh} \arrow[d, dotted, "\exists! \text{ by } sh \text{ univ. prop.}"] \\
 & & \mathcal{I} \text{ (a sheaf)}
\end{tikzcd}
$$

Another perspective:

$$
\begin{array}{ccc}
\operatorname{Hom}_{Ab_X}(\mathcal{H}^{sh}, \mathcal{I}) & \overset{\text{want}}{\cong} & \ker\left(\operatorname{Hom}_{Ab_X}(\mathcal{G}, \mathcal{I}) \to \operatorname{Hom}_{Ab_X}(\mathcal{F}, \mathcal{I})\right) \\
\wr\!\wr\ sh & & \Vert \\
\operatorname{Hom}_{Ab_X^{pre}}(\mathcal{H}, \mathcal{I}) & \underset{\operatorname{coker}^{pre}}{\cong} & \ker\left(\operatorname{Hom}_{Ab_X^{pre}}(\mathcal{G}, \mathcal{I}) \to \operatorname{Hom}_{Ab_X^{pre}}(\mathcal{F}, \mathcal{I})\right)
\end{array}
$$

Thm: Ab_X is an abelian category.

Two main ingredients to the proof:

(1) Sheafifications exist (hence cokernels exist)

(2) "Exactness can be checked on stalks"] + Ab is an abelian category

$\Uparrow$

isomorphisms can be checked on the level of stalks.

For both ingredients: useful to have def of sheaves in terms of stalks.

<u>Def</u> (of a sheaf, again): A sheaf $\mathcal{F}$ on X is a presheaf s.t. for any open $U \subseteq X$, the map

$$\alpha : \mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p$$ is injective and has image described by the "compatible germs" condition:

$(s_p \in \mathcal{F}_p)_{p \in U}$ is in the image of α if and only if if $q \in U$, then there exists $\underset{q}{\underset{\rotatebox{90}{$\in$}}{V}} \overset{\text{open}}{\subseteq} U$ and $\underbrace{s_V \in \mathcal{F}(V)}$

s.t. $(s_V)_p = s_p$ for all $p \in V$.

Then the sheafification $\mathcal{F}^{sh}$ of a presheaf $\mathcal{F}$ can be constructed by

$$\mathcal{F}^{sh}(U) = \left\{ (s_p) \in \prod_{p \in U} \mathcal{F}_p \;\middle|\; \begin{array}{c}\text{compatible germs} \\ \text{condition for } \mathcal{F}\end{array} \right\}$$

(Need to check: $\mathcal{F}^{sh}_p \cong \mathcal{F}_p$, compatible germs conditions for $\mathcal{F}$ and $\mathcal{F}^{sh}$ are equiv.)

Idea here: hold stalks fixed, force presheaf to become a sheaf.

Can check:

$$\operatorname{coker}\left(f : \mathcal{M}_{\mathbb{Z}} \to \mathcal{M}_{\mathbb{R}}\right) \cong \mathcal{M}_{\mathbb{R}/\mathbb{Z}}$$

$$\operatorname{coker}\left(g : \mathcal{M}_{\mathbb{R}} \to \mathcal{M}_{\mathbb{R}/\mathbb{Z}}\right) \cong 0.$$

Lemma: $\pi : \mathcal{F} \to \mathcal{G}$ is an isomorphism ($\mathcal{F}, \mathcal{G} \in \mathrm{Sets}_X$)

$\iff \pi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for all p.

Pf: $\Rightarrow$ is immediate.

$\Leftarrow$: Want to construct $\pi^{-1} : \mathcal{G} \to \mathcal{F}$.

Given $s \in \mathcal{G}(U)$, $(s_p)_{p \in U}$ are compatible.

Want to check that $(\pi_p^{-1}(s_p))_{p \in U}$ are also compatible.

Suppose $q \in U$ and take $(V, t \in \mathcal{F}(V)) = \pi_q^{-1}(s_q) \in \mathcal{F}_q$ for $q \in V \subseteq U$.

Then $\pi(V)(t)$ and $s|_V$ have the same germ s_q at q, so they agree on some $W \overset{\text{open}}{\subseteq} V$, $W \ni q$.

But then $\pi(W)(t|_W) = s|_W$, so they have the same germs for all $p \in W$.

So $(t|_W)_p = \pi_p^{-1}(s_p)$ for all $p \in W$.

So there is some section in $\mathcal{F}(U)$ with germs $\pi_p^{-1}(s_p)$ at each $p \in U$. Call this $\pi^{-1}(U)(s)$, and define a morphism $\pi^{-1}: \mathcal{G} \to \mathcal{F}$ in this way. Since $(\pi^{-1})_p = (\pi_p)^{-1}$, $\pi \circ \pi^{-1}$ and $\pi^{-1} \circ \pi$ both induce the identity map on all stalks, and hence $\pi \circ \pi^{-1} = id_{\mathcal{G}}$, $\pi^{-1} \circ \pi = id_{\mathcal{F}}$.

$\blacksquare$