- new office hour times: M: 11-12

  W: 2:30-3:30

  F: 4-5

- first problem set posted Thursday (will say more on Thursday)
- put Lecture note pdfs on the main website.

Last time: stalks: $\mathcal{F}_p = \{(U \ni p, s \in \mathcal{F}(U))\}/\sim$

<u>Lemma:</u> (germs determine sections): Let $\mathcal{F}$ be a sheaf on X, let $U \underset{\text{open}}{\subseteq} X$. Then the natural map:

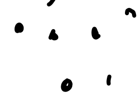
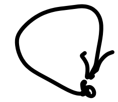
$$\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p \text{ is injective.}$$

<u>Pf:</u> If s_1, s_2 agree on all stalks, then they agree on an open cover, so $s_1 = s_2$ by identity axiom for sheaves.

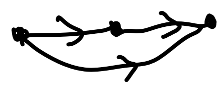
Morphisms of sheaves:

Want sheaves to form a category

(Category = objects, morphisms, identity morphisms,

composition of morphisms)



Tempting to define a morphism from a sheaf $\mathcal{F}$ on X to a sheaf $\mathcal{G}$ on Y as an underlying continuous map $X \to Y$ + something on the sheaf structure.

(We won't do this, but the category of schemes will be defined similarly later.)

Instead: fix X, consider two sheaves on X.

Def: Let X be a top. space. The category of sheaves (of sets) on X is denoted Sets_X.

The objects of Sets_X are sheaves $\mathcal{F}$ on X.

The morphisms of Sets_X are defined as follows:

$\mathcal{F}, \mathcal{G} \in \text{Sets}_X$. A morphism $\pi : \mathcal{F} \to \mathcal{G}$ is given by: $\pi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ (functions of sets)

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "\mathrm{res}_{UV}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U) \arrow[d, "\mathrm{res}_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\pi(V)"'] & \mathcal{G}(V)
\end{tikzcd}$$

commuting with restriction maps.

Similarly define Sets_X^{pre} to be the category of presheaves on X.

<u>Example:</u> X, Y top. spaces, $p \in X$

$\mathcal{F}$ = sheaf on X of continuous functions to Y

$\mathcal{G}$ = skyscraper sheaf on X at $p \in X$ with values in Y.

$\pi: \mathcal{F} \to \mathcal{G}$ morphism given by taking value of function at p.

Some of our definitions/constructions from last time can be interpreted as functors:

pushforward: if $\pi: X \to Y$ is continuous, can define a functor $\pi_*: \text{Sets}_X \to \text{Sets}_Y$.

(functor: take objects to objects,
morphisms to morphisms,
distributes over composition of morphisms,

so here we need an induced map
$\pi_* \mathcal{F} \to$

stalk at a given point: $(\ )_p : \text{Sets}_X \to \text{Sets}$.
~~(these~~ (morphisms of sheaves induce maps on stalks)

Lemma: (morphisms are determined by stalks)

$$\underbrace{\mathrm{Mor}_{\mathrm{Sets}_X}}_{\text{set of morphisms in category } \mathrm{Sets}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \prod_{p \in X} \mathrm{Mor}_{\mathrm{Sets}}(\mathcal{F}_p, \mathcal{G}_p)$$

is injective.

Pf. (consequence of previous lemma).

In practice, our sets of sections $\mathcal{F}(U)$ will usually have more structure.

Def: The category of sheaves of abelian groups on X is denoted Ab_X and is identical to Sets_X except that the $\mathcal{F}(U)$ and restriction/sheaf morphism maps between them are in $\underbrace{Ab}$.

important that this is a category of "sets with extra structure"

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \end{array}$$

Similarly define Ab_X^{pre}, Rings_X, Rings_X^{pre}, ...

In nice cases, stalks will exist and have the appropriate structure themselves:

if $\mathcal{F} \in Ab_X$, then $\mathcal{F}_p \in Ab$ $\qquad \{(U,s)\}/\sim$

The type of sheaves that will matter most to us is sheaves of rings.

It will be useful to take a digression to Ab_X.

<u>Thm</u>: 1) Ab_X^{pre} is an abelian category

2) Ab_X is $\underbrace{\text{an abelian category}}_{\text{category like } Ab,\ Mod_A}$

Today: define abelian category, part 1 of Thm (presheaves)

Thursday: part 2 (sheaves)

Def: A category $\mathcal{C}$ is <u>additive</u> if:

- for $A, B \in \mathcal{C}$, $Mor(A, B)$ has the structure of an abelian group, and composition distributes over ~~ad~~ addition.
- There is a zero object $0 \in \mathcal{C}$, i.e. $$Mor(A, 0) = Mor(0, A) = 0.$$
- products of two objects exist. ($\underbrace{\text{products}}_{\text{categorical}}$)

(in an additive category, write $Hom(A, B)$ instead of $Mor(A, B)$)

Def: An <u>abelian category</u> is an additive category satisfying:

- <u>kernels</u> and <u>cokernels</u> exist
- every monomorphism is the kernel of its cokernel
- every epimorphism is the cokernel of its kernel.

Def: The <u>product</u> of two objects $A, B \in \mathcal{C}$ is an object $A \times B \in \mathcal{C}$ along with morphisms to A and B s.t. if D is another such (object + morphisms to A and B), then it factors uniquely through $A \times B$!

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & A \\ B \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & A \times B \arrow[r] \arrow[d] & A \\ & B & \end{tikzcd}$$

exists and is unique such that both triangles commute.

If a product $A \times B$ exists in $\mathcal{C}$, it is "unique up to unique isomorphism"

Fact about universal prop. definitions like this:

$A \times B$ is unique up to unique isomorphism;

If X_1, X_2 were two candidates for $A \times B$:

$$\begin{tikzcd} X_1 \arrow[dr, shift left, "2"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X_2 \arrow[ul, shift left] \arrow[r] \arrow[d] & A \\ & B & \end{tikzcd}$$

<u>Def</u>: The <u>kernel</u> of a morphism $\pi: A \to B$ in an additive category $\mathcal{C}$ is an object $(\ker \pi) \in \mathcal{C}$ along with a morphism $\tau: (\ker \pi) \to A$ s.t. $\pi \circ \tau : (\ker \pi) \to B$ is the zero morphism, and s.t. if D, $p: D \to A$ satisfy the same condition $\left((\pi \circ p) = 0 \in \mathrm{Hom}(D, B) \right)$, then (D, p) factors uniquely through $(\ker \pi, \tau)$:

$$\begin{tikzcd}
D \arrow[r, dashed, "\exists!"'] \arrow[dr, bend right, "p"'] \arrow[drr, bend left=60, "0"] & \ker \pi \arrow[d, "\tau"] \arrow[dr, "0"] & \\
& A \arrow[r, "\pi"] & B
\end{tikzcd}$$

Can define a cokernel similarly using.

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[dr, "0"'] \arrow[ddrr, bend right=60, "0"'] & B \arrow[d, "\tau"] \arrow[ddr, bend left, "p"] & \\
& \operatorname{coker} \pi \arrow[dr, dashed, "\exists!"'] & \\
& & D
\end{tikzcd}$$

Note: in general, given a $\underbrace{\text{diagram}}_{\text{collection of objects and morphisms between them}}$ in a category $\mathcal{C}$,

can define the limit of the diagram as the universal object mapping to the diagram.

product: diagram

$$\begin{array}{cc} & A \\ B & \end{array}$$

(no arrows)

limit: universal object

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & A \\ B & \end{tikzcd}$$

diagram

kernel: diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r, "\pi"'] & B \end{tikzcd}$$

$$\begin{tikzcd} \ker \pi \arrow[r] \arrow[d] \arrow[dr, "0"] \arrow[dr, bend left=60] \arrow[dr, bend right=60, "\text{composition}"'] & 0 \arrow[d] \\ A \arrow[r, "\pi"'] & B \end{tikzcd}$$