Thanks to those who filled out the survey
– if you haven't yet, please do so.

Today: definition of a sheaf, many examples, stalks

<u>Def</u>: A presheaf (of sets) $\mathcal{F}$ on a top. space X is the following data:

- a set $\mathcal{F}(U)$ for each $U \overset{\text{open}}{\subseteq} X$
- a restriction map $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for each $\underset{\text{open}}{V \subseteq} \underset{\text{open}}{U \subseteq} X$

satisfying $r_{VW} r_{UV} = r_{UW}$ and $r_{UU} = id_{\mathcal{F}(U)}$.

Examples:

1) constant presheaf $\mathcal{F}(U) = S$
$r_{UV} = id_S$
for some fixed S.

2) Let Y be a top. space. $U \underset{\text{open}}{\subseteq} X$

Then take $\mathcal{F}(U) = \{ f : U \xrightarrow[\text{continuous}]{} Y \}$,

and define r_{UV} by restriction of functions.

<u>Notation:</u>

1) elements $s \in \mathcal{F}(U)$ are called <u>sections</u> of $\mathcal{F}$ (over U)

2) $r_{UV}(s)$ is often denoted $s|_V$

3) $\boxed{\mathcal{F}(U)}$ is sometimes denoted

$$\Gamma(U, \mathcal{F}) \ \underline{\underline{\text{or}}} \ H^0(U, \mathcal{F})$$

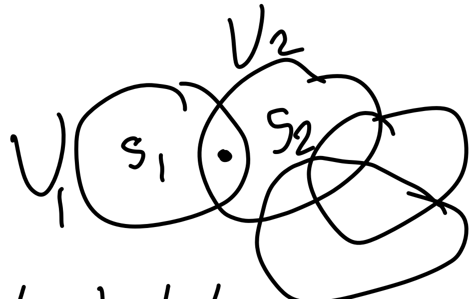
$$\Gamma(U,\mathcal{F}) = \mathcal{F}(U) = H^0(U,\mathcal{F})$$

Sheaf def 1:

Def: A sheaf (of sets) $\mathcal{F}$ on X is a presheaf satisfying:

identity axiom: Let $U \overset{\text{open}}{\subseteq} X$, let $\{U_i\}_{i\in I}$ be an open cover of U, and suppose $s_1, s_2 \in \mathcal{F}(U)$. Then

$$s_1 = s_2 \iff s_1|_{U_i} = s_2|_{U_i} \text{ for all } i.$$



gluability axiom: Let $U \overset{\text{open}}{\subseteq} X$, let $\{U_i\}$ be an open cover of U, and suppose $s_i \in \mathcal{F}(U_i)$ for each i. Further suppose $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there exists $\underbrace{s \in \mathcal{F}(U)}_{\text{unique}}$ with $s|_{U_i} = s_i$ for each i.

<u>Sheaf def 2:</u> A sheaf $\mathcal{F}$ on X is a presheaf satisfying:

Let $U \subseteq X$ be open, let $\{U_i\}$ be an open cover of U, and suppose $s_i \in \mathcal{F}(U_i)$ agree on double intersections. Then there $\underbrace{\text{exists}}$ a $\underbrace{\text{unique}}$ $s \in \mathcal{F}(U)$ extending all of them $(s|_{U_i} = s_i)$

gluability identity.

(just a restatement of def 1)

<u>Note:</u> a sheaf is a flexible way to describe additional structure on a top. space.

The sheaf axioms mean that this additional structure can be described on small open sets.

Earlier examples: $r_{UV} = \mathrm{id}_S$

- constant ~~sheaf~~ presheaf $\mathcal{F}(U) = S$ fails both axioms in general.

$S \neq S \times S$

$U = U_1 \sqcup U_2$

doesn't actually fail until $\mathcal{F}(\phi) = \{\cdot\}$.

Gluability fails if $|S| > 1$ and there is some disconnected open set.

Identity fails because of the empty open cover of the empty set:

$$\phi = \bigcup_{i \in \phi} V_i,$$ i.e. $\mathcal{F}(\phi) = S$ is not allowed if $|S| > 1$.

(Can see by this argument: $\mathcal{F}(\phi) = \{\cdot\}$ for any sheaf $\mathcal{F}$).

Let's fix the constant presheaf to make it a sheaf:

$$\mathcal{F}(U) = \begin{cases} S & \text{if } U \neq \phi \\ \{\cdot\} & \text{if } U = \phi \end{cases} \Leftrightarrow \mathcal{F}(U) = \{\text{constant functions } U \to S\}$$

This fixes identity axiom but not gluability.

Def/Example: The <u>constant sheaf</u> with values in S, denoted $\underline{S}$, is given by

$$\underline{S}(U) = \left\{ \begin{array}{c} \text{locally constant functions} \\ U \longrightarrow S \end{array} \right\}.$$

U_1 $\quad$ U_2

(same thing as continuous functions $U \to S$ if S has the discrete topology)

- $\mathcal{F}(U) = \{ f : U \xrightarrow[\text{cont.}]{} Y \}$ is a sheaf.
  - still a sheaf with continuous replaced by differentiable, smooth, holomorphic, ...
  - but fails "existence" of gluing for bounded functions (works for "locally bounded")

More examples:

1) restriction to open subset.

If $\mathcal{F}$ is a sheaf on X and $V \underset{\text{open}}{\subseteq} X$, then can define $\mathcal{F}|_V$ as a sheaf on V by

$$\mathcal{F}|_V(U) = \mathcal{F}(U) \text{ for } U \underset{\text{open}}{\subseteq} V \subseteq X.$$

2) Def: Let $\pi: X \to Y$ be continuous and let $\mathcal{F}$ be a sheaf on X. Then the pushforward sheaf $\pi_* \mathcal{F}$ on Y is given by $(\pi_* \mathcal{F})(U) = \mathcal{F}(\pi^{-1}(U))$ (and obvious restriction maps)

(Why is this a sheaf? if $\{U_i\}$ is an open cover of U, then $\{\pi^{-1}(U_i)\}$ are an open cover of $\pi^{-1}(U)$)

Special case of pushforward sheaf:

Y top. space

$p \in Y$ point

S is a set.

$$X = \{\cdot\}$$

$$\pi : \{\cdot\} \longrightarrow Y$$

$$\cdot \longmapsto p.$$

$\mathcal{F}$ = constant sheaf $\underline{S}$ on $\{\cdot\}$

$\pi_* \mathcal{F}$ should be a sheaf on Y encoding the choice of S and of $p \in Y$.

$$(\pi_* \mathcal{F})(U) = \begin{cases} S & \text{if } p \in U \\ \{\cdot\} & \text{otherwise} \end{cases}$$

"skyscraper sheaf".

Pushforward is very flexible: another interesting example:

$\pi : X \to Y$ is some covering space,

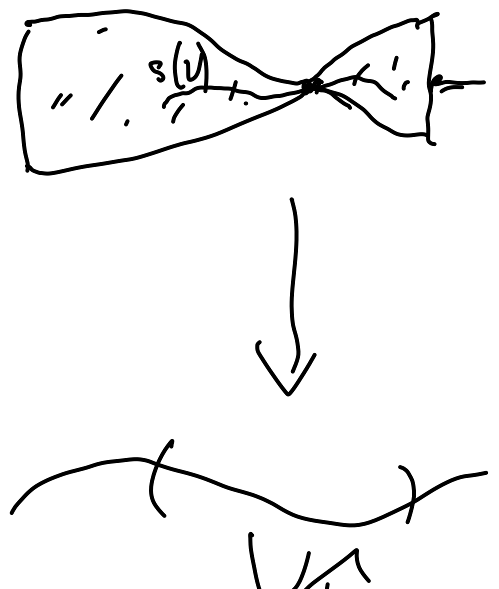
$\mathcal{F}$ on X is the sheaf of continuous functions to $\mathbb{Z}$.

Then $\pi_* \mathcal{F}$ contains some of the structure of π.

Final example of a sheaf:

Let $\pi : Y \to X$ be continuous. Define a sheaf $\mathcal{F}$ on X by

$$\mathcal{F}(U) = \left\{ \begin{array}{l} \text{continuous sections of } \pi \text{ over } U, \\ \text{i.e. } s : U \to Y \text{ s.t. } \begin{tikzcd} U \arrow[r, "s"] \arrow[dr, "\iota"'] & Y \arrow[d, "\pi"] \\ & X \end{tikzcd} \\ \text{commutes,} \\ \text{i.e. } \pi \circ s = \iota \\ \pi(s(u)) = u \text{ for all } u \in U \end{array} \right\}$$



$$\begin{tikzcd} Y \arrow[d, "\pi"'] \\ X \arrow[u, bend right, "s"'] \end{tikzcd}$$

Special case:

$Y = X \times Z$, $\pi = pr_1 : X \times Z \to X$,

get $\mathcal{F}(U) = \{\text{cont. functions } U \to Z\}$

Skyscraper example:

$Y = X \setminus \{p\} \sqcup S$

with discrete topology

<u>Fact</u>: Any sheaf $\mathcal{F}$ on X is isomorphic to the sheaf of sections of some continuous map $\pi: Y \to X$.

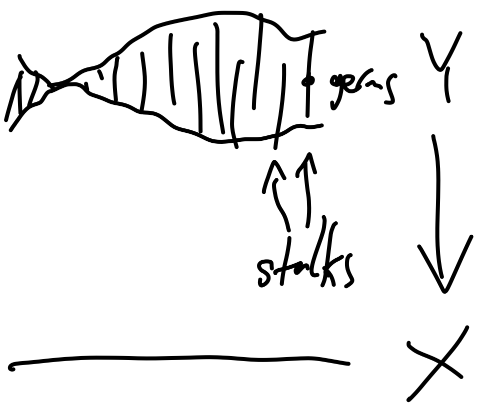
Key idea for understanding sheaves: <u>stalks</u>

<u>Def</u>: Let $\mathcal{F}$ be a sheaf on X, and let $p \in X$. Then the <u>stalk of $\mathcal{F}$ at p</u>, denoted $\mathcal{F}_p$, is

$$\mathcal{F}_p := \left\{ (U, s) \,\middle|\, \begin{array}{c} p \in U \overset{\text{open}}{\subseteq} X \\ s \in \mathcal{F}(U) \end{array} \right\} \Big/ \sim ,$$

where $\sim$ is the equivalence relation generated by $(U, s) \sim (V, s|_V)$ for $p \in V \underset{\text{open}}{\subseteq} U$.

Elements of $\mathcal{F}_p$ are called germs.



Example: If $\mathcal{F}$ is the skyscraper sheaf with set S at point $p \in X$, then

$$\mathcal{F}_q = \begin{cases} S & \text{if } q \in \overline{\{p\}} \\ \{\cdot\} & \text{else} \end{cases}$$

Next time: - "germs determine sections"
- turn sheaves into a category Sets_X (morphism of sheaves)
- sheaves of ab. groups, rings, etc.