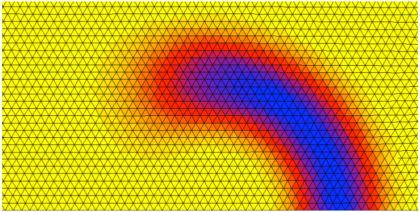
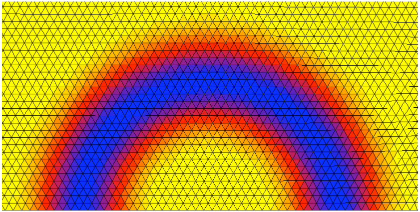
Taylor-Galerkin Residual Distribution Schemes with Application to Astrophysical Flows

James A. Rossmanith

Department of Mathematics
University of Wisconsin – Madison

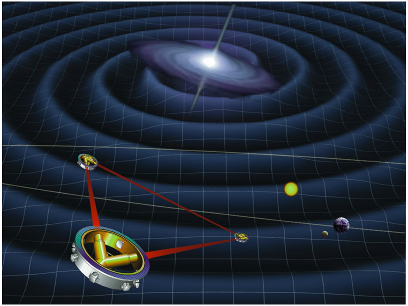




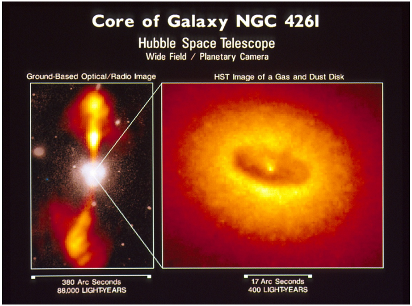
REU Student: Matt Elsey, U. Michigan – developed grid generator in C++

Motivation

lisa.jpl.nasa.gov



hubblesite.org



- Astrophysical fluids: stellar collapse, interaction of BHs, BH accretion, . . .
- Numerical challenges: shocks, geometric singularities, constraints, . . .
- Numerical methods: FD, spectral, SPH, HRSC (Godunov, ENO, & central)
- Goal of this research: develop alternative based on residual distribution
- Properties: unstructured & truly multi-D $\Longrightarrow$ high-order & unsteady under research
- This talk: basic idea & preliminary results on simple equations

Residual distribution schemes

Basic idea: [Roe, 1987], [Deconinck et al., 1993]

- Hyperbolic balance law: $\partial_t q + \vec{\nabla} \cdot \vec{F} = \psi$
- Solution is stored on nodes of a triangular (tetrahedral) mesh
- Create residual in each element & distribute to nodes:

$$\phi^{\mathcal{T}} \approx \iint_{\mathcal{T}} \left[\vec{\nabla} \cdot \vec{F} - \psi \right] \, dA \quad \Longrightarrow \quad \phi_1^{\mathcal{T}}, \, \phi_2^{\mathcal{T}}, \, \phi_3^{\mathcal{T}}$$

- Update solution by collecting all residuals that influence node i:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{|C_i|} \sum_{T : i \in T} \phi_i^{\mathcal{T}}$$

Why RD schemes for astrophysics?

- Shock-capturing, high-order on smooth flows
- Can handle complicated, even time-dependent geometry
- RD schemes are well-balanced $\Longrightarrow$ steady-states are accurately preserved

1D Residual distribution schemes

Note: For $q_t + f(q)_x = 0 \implies$ RD schemes $\equiv$ HRSC (Roe's method)

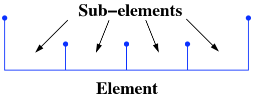
Distribution point interpretation of limiters: $(q_t + u q_x = 0)$

$$\phi_1 = \left(\frac{x_i - p}{x_i - x_{i-1}} \right) \phi^{\mathcal{T}} \quad \text{and} \quad \phi_2 = \left(\frac{p - x_{i-1}}{x_i - x_{i-1}} \right) \phi^{\mathcal{T}}, \quad \text{where} \;\; p \in [x_{i-1}, x_i]$$

- **N-Scheme:** upwind method is "narrow" $\implies$ either $p = x_i$ or $p = x_{i-1}$
- **Lax-Wendroff:** $p = x_{i-1/2} + \frac{1}{2} \Delta x \, \nu$, where $-1 \le \nu = \frac{u \Delta t}{\Delta x} \le 1$
- **Limiters:** Find p s.t. solution in non-oscillatory $\implies$ if $u > 0$: $p_{\text{LxW}} \le p \le p_{\text{N}}$

Higher-order for 1D steady-states:

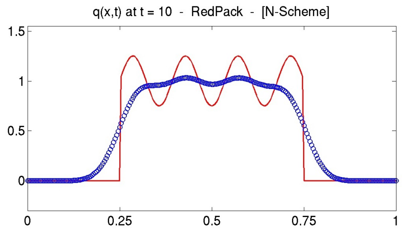
- Create sub-elements in each element, solution stored at each node

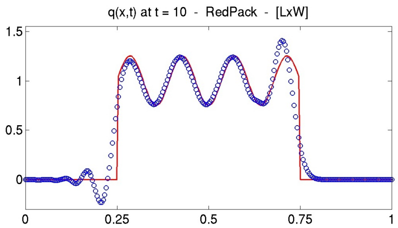


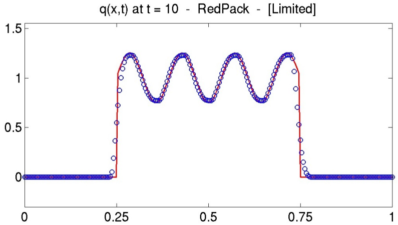
$$\phi^{\mathcal{T}} = f_i - f_{i-1} - \int_{x_{i-1}}^{x_i} \psi \, dx$$

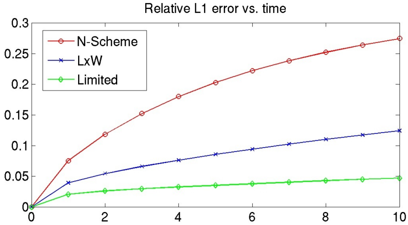
- In each element create interpolant of ψ, integrate exactly in each sub-element

Example: 1D periodic advection









Application: relativistic gas dynamics

$$\frac{\partial}{\partial x^0}\left(\sqrt{-g}\begin{bmatrix} D \\ S^j \\ E \end{bmatrix}\right) + \frac{\partial}{\partial x^i}\left(\sqrt{-g}\begin{bmatrix} \rho v^i W \\ \rho h v^i v^j W^2 + p g^{ij} \\ \rho h v^i W^2 + p g^{i0} \end{bmatrix}\right) = -\begin{bmatrix} 0 \\ \sqrt{-g}\,\Gamma^i_{\mu\lambda} T^{\mu\lambda} \\ \sqrt{-g}\,\Gamma^0_{\mu\lambda} T^{\mu\lambda} \end{bmatrix}$$

Covariant formulation [Papadopolous & Font, 2000]

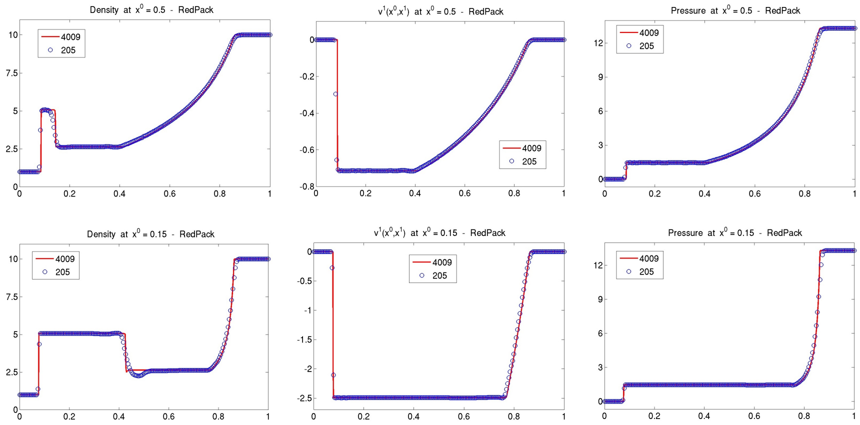
- $q = (D, S^j, E) \equiv$ (rest-mass, momentum, energy)
- $u = (\rho, v^j, p) \equiv$ (density, fluid 3-velocity, fluid pressure)
- Specific relativistic enthalpy: $h = 1 + \frac{p\Gamma}{\rho(\Gamma-1)}$, $\Gamma \equiv$ gas constant
- Relationship between conserved and primitive variables

$$\begin{bmatrix} D \\ S^j \\ E \end{bmatrix} = \begin{bmatrix} \rho W \\ \rho h v^j W^2 + p g^{0j} \\ \rho h W^2 + p g^{00} \end{bmatrix}, \qquad W = \frac{1}{\sqrt{-g_{00} - 2 g_{0i} v^i - g_{ij} v^i v^j}}$$

Choice of spacetime foliation

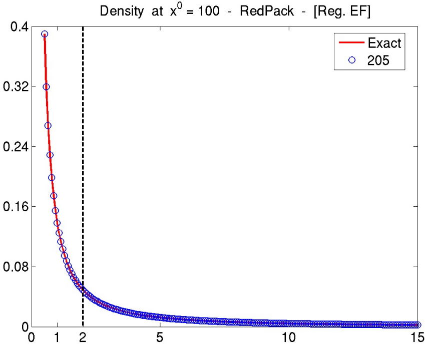
- Spacelike foliations of spacetime ($g^{00} \neq 0$) $\Longrightarrow$ Newton iteration for W
- Null foliations of spacetime ($g^{00} = 0$) $\Longrightarrow$ no Newton iteration required

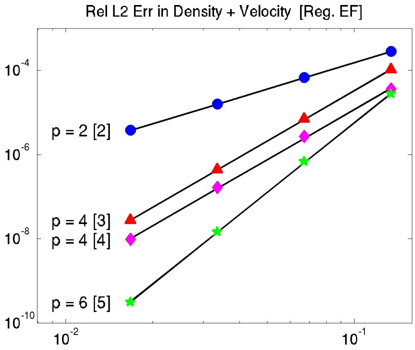
Example: special relativistic hydrodynamics



- **Example 1:** Minkowski coordinates $(t, x) = (x^0, x^1)$, $\quad ds^2 = -dx^0 dx^0 + dx^1 dx^1$
- **Example 2:** Null coordinates $(t, x) = (x^0 - x^1, x^1)$, $\quad ds^2 = -dx^0 dx^0 + 2dx^0 dx^1$

Example: radial dust accretion



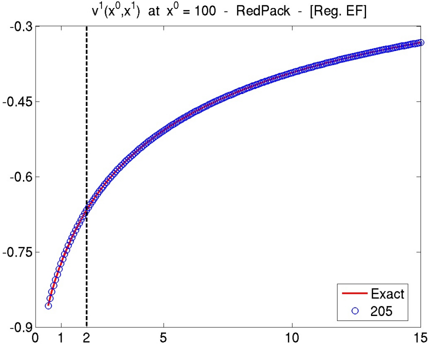


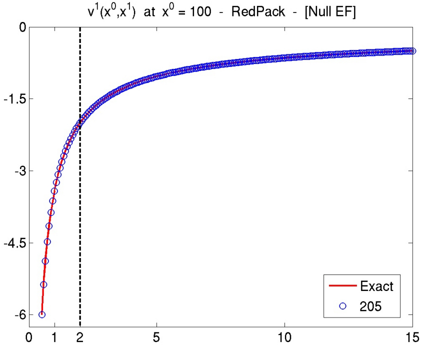
- Eddington-Finkelstein coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\hat{t}^2 + \frac{4M}{r} d\hat{t}\, dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

- Start from constant density state with $p \approx 0$, run to steady-state
- Exact solution known, experimental convergence rate: 2, 4, 4, and 6

Example: radial dust accretion



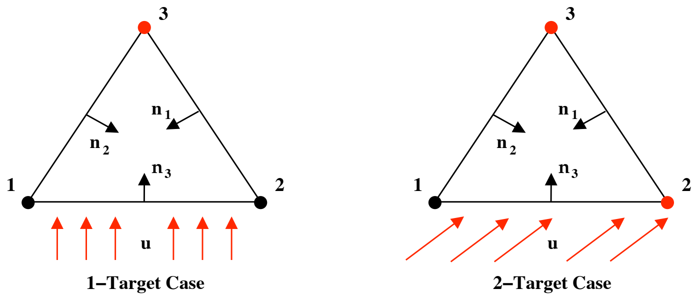


- Null Eddington-Finkelstein coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\hat{t}^2 + 2\, d\hat{t}\, dr + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

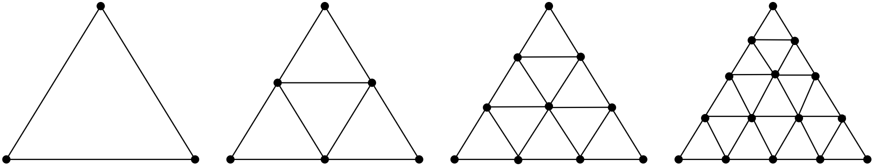
- Start from constant density state with $p \approx 0$, run to steady-state
- Exact solution known

2D N-scheme



- Narrow scheme: minimal stencil for 1^{st} order, 2D upwind method
- Residual: $\phi_i = \frac{1}{2}\left[\vec{u}\cdot\vec{n}_i\right]^+ (Q_i - \bar{Q}) \equiv \beta_i \phi^{\mathcal{T}}$
- **Conservation:** $\bar{Q}$ chosen so that $\sum_i \phi_i = \phi^{\mathcal{T}} \implies \sum_i \beta_i = 1$
- **Monotonicity:** N-scheme can be written as $\phi_i = \sum_i \sum_j c_{ij}(Q_i - Q_j)$, $c_{ij} \geq 0$
- **Linear Preserving:** improved accuracy in steady-state with $\beta_j \to \frac{\beta_j^+}{\sum_j \beta_j^+}$

2D High-order N-scheme



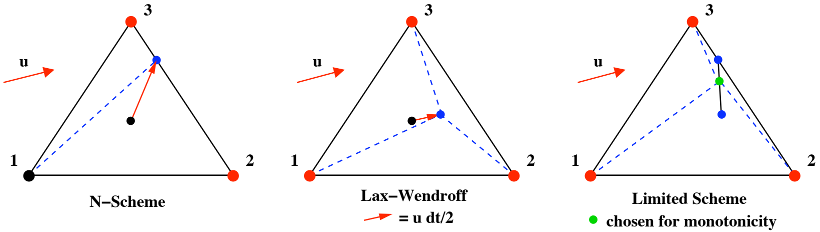
2nd Order Element | 3rd Order Element | 4th Order Element | 5th Order Element

- **[Abgrall & Roe, 2003]** – high-order in steady-state using above elements
- $P_k(x, y) \equiv$ interpolant over full element
- Construct a high-order residual by integrating $P_k(x, y)$ in each sub-element $\mathcal{T}$:

$$\phi^{\mathcal{T}} = \iint_{\mathcal{T}} P_k(x, y)\, dA \approx \iint_{\mathcal{T}} \left[\vec{\nabla} \cdot \vec{F} - \psi \right] dA$$

- In practice this is carried out using 2D Gaussian quadrature
- Distribute via the N-scheme in each sub-element
- To achieve high-order need to again apply: $\beta_j \longrightarrow \frac{\beta_j^+}{\sum_j \beta_j^+}$

2D Lax-Wendroff scheme and limiting

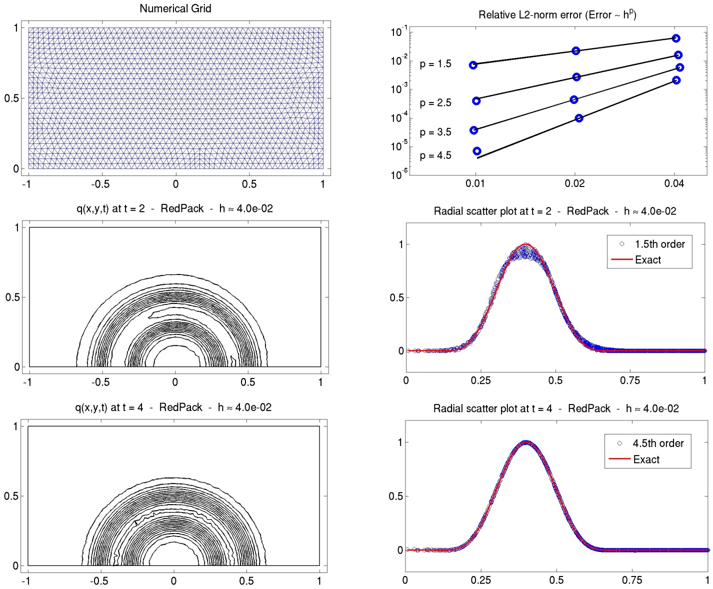


- **[Hubbard & Roe, 2000]** – Limiters on N-scheme + Lax-Wendroff
- Lax-Wendroff method:

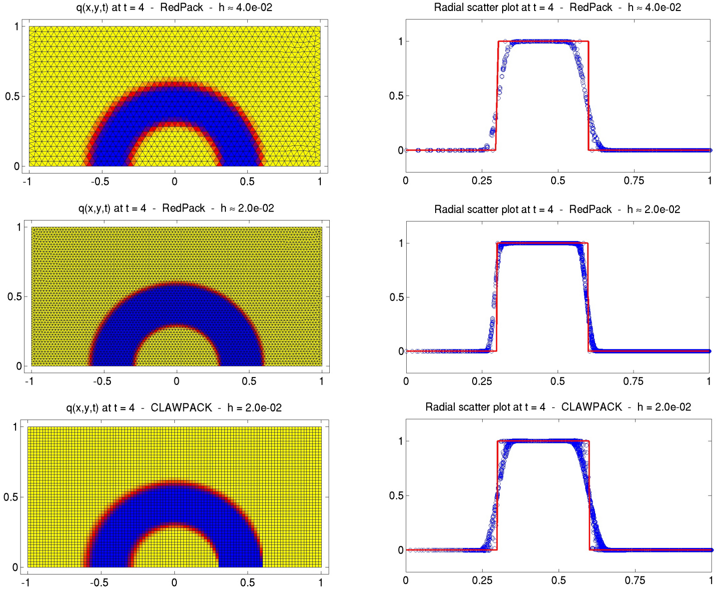
$$\phi_i = \left(\frac{1}{3} + \frac{\Delta t}{4 S_{\mathcal{T}}} \left[\vec{u} \cdot \vec{n}_i \right] \right) \phi^{\mathcal{T}}$$

- Limiting: N-scheme is monotone, so don't allow values to exceed N-scheme values
- **Limiting for systems:** open problem
- **Higher-order:** open, work by **[Abgrall et al., 2005]** and **[Hubbard & Laird, 2005]**

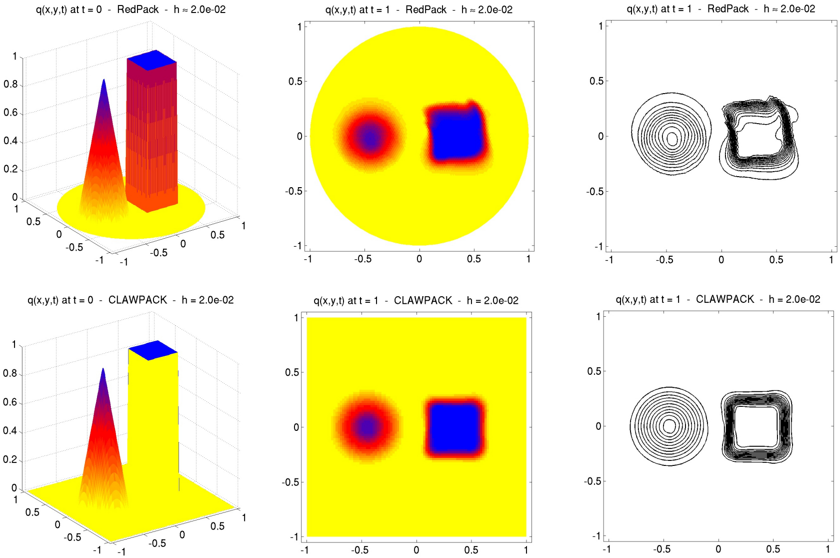
Example: 2D steady advection (smooth)



Example: 2D steady advection (discontinuous)



Example: 2D unsteady advection



Future work

1. Systems of conservation laws
2. Improve time accuracy
3. Develop 3D code (3D mesh generation already operational)
4. Relativistic Euler & magnetohydrodynamics
5. Simulation of black hole accretion
6. Dynamically evolving spacetimes