Math 471-001 Fall 2013 Computing Project 1 due : Tuesday, Oct. 22

Project A : Differentiation matrices

In homework 1 we saw the 2nd-order accurate centered difference approximation

$$f'(x) \approx D_0 f(x) = \frac{f(x+h) - f(x-h)}{2h},$$

which we used to approximate $f'(x)$ at a specific point $x = x_0$. In this problem we consider the similar problem of needing to approximate $f'(x)$ across a range of x values associated with a spatial domain.

1. Consider the 2π-periodic function $f(x) = e^{\sin x}$. Discretize the interval $[0, 4\pi)$ by $x_n = \{x_1, x_2, \dots, x_{n-1}, x_n\}$ into n equal subintervals of length $h = 4\pi/n$, with $x_1 = 0$ and $x_n = 4\pi - h$. Let w_i stand for the approximation of $f'(x)$ at $x = x_i$. Apply the centered difference formula to find $w_i \approx f'(x_i)$ (hint: since $f(x)$ is periodic, $f(x_0) = f(x_n)$ and $f(x_{n+1}) = f(x_1)$):

$$\begin{aligned} w_1 &= \frac{1}{2h}(f(x_2) - f(x_n)) \\ w_2 &= \frac{1}{2h}(f(x_3) - f(x_1)) \\ \vdots &= \ \vdots \\ w_n &= \frac{1}{2h}(f(x_1) - f(x_{n-1})) \end{aligned}$$

2. Let $\boldsymbol{y} = [f(x_1), f(x_2), \dots, f(x_n)]^T$ and $\boldsymbol{w} = [w_1, w_2, \dots, w_n]^T$. Show that the system of equations in part (a) can be written as $\boldsymbol{w} = D_0 \boldsymbol{y}$, where the matrix D_0 is

$$D_0 = \frac{1}{2h} \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & & & -1 & 0 & 1 & & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{bmatrix}.$$

3. For $f(x) = e^{\sin x}$, plot the graph of $\boldsymbol{w}$ and the exact derivative $f'(x) = \cos x e^{\sin x}$ on the interval $x \in [0, 4\pi)$ using $n = 10,\ 20$, and 80. Notice that most of the entries in D_0 are zero; thus, D_0 is an example of a *sparse matrix*. Use the Matlab tool `diag` to construct D_0.

4. Repeat part 4 for the following functions on the interval $[0, 10\pi)$.

   (a) $g(x) = \sin x + 2 \sin 3x \cos x$

   (b) $h(x) = \sin x + \sin 10x$

(c) $y(x) = e^{\sin x \cos x}$

5. The matrix D_0 approximates the first derivative of a function given n values of that function; it seems reasonable to expect that $(D_0)^2\boldsymbol{y} = D_0 D_0 \boldsymbol{y}$ might approximate the second derivative. Test this idea on the function $f(x) = e^{\sin x}$. Compare the structure of $(D_0)^2$ to D_0 and comment on the number of nonzero diagonals in $(D_0)^2$ compared to D_0.

6. Explore the connection between the operation of matrix multiplication and differentiation further by considering $(D_0)^n$ as a proxy for $f^{(n)}$, the nth derivative of a function. Find the first 4 derivatives of $f(x) = \sin x + 0.3 \sin 2x - 0.4 \sin 3x$. Write a Matlab program that computes the first four derivatives' approximations $(D_0)^k f(x)$ for $k = 2, 3, 4$ and $x \in [0, 6\pi)$. Plot the exact derivatives on the same axes with their approximations using $n = 100$ grid points. Discuss the connection between the step size h and the accuracy of the approximation as the order of differentiation increases (note: this will require several values of n). Support your discussion with quantitative results (tables, Taylor series analysis, or log-log plots).

7. Let the differentiation matrix D_N denote the matrix D_0 for a given N, the number of subintervals in the domain $x \in [0, 1)$. Explore the eigenvalues of D_N as a function of N, where $N = 2^k$ for $k = 3, 4, ..., 12$. Do any patterns emerge? For example, is D_N nonsingular for any N? Are there any eigenvalues of D_N with nonzero real parts? What range do the eigenvalues fall into? Do they appear to be uniformly distributed in that interval? (Hint : examine the Matlab commands `eigs` and `eig`.)

Project B : Two-point boundary value problems

Use the methods discussed in class for the finite difference scheme and LU solver. Note that your code should not construct the entire matrix; instead, use linear arrays containing the nonzero matrix elements.

1. Consider the 2-point BVP, $-\epsilon y'' + y = 2x + 1$ on the domain $0 \le x \le 1$, with boundary conditions $y(0) = 0,\ y(1) = 0$ and $\epsilon = 10^{-3}$.

   Show that the exact solution is $y(x) = 2x+1-\left(\sinh \frac{1-x}{\sqrt{\epsilon}} + 3 \sinh \frac{x}{\sqrt{\epsilon}}\right)\left(\sinh \frac{1}{\epsilon}\right)^{-1}$.

   Plot the exact solution and the numerical solution for $h = \frac{1}{32}$.

2. Consider the 2-point BVP
$$\epsilon y'' + y = f(x)$$
   on $x \in [0, 1]$ with $y(0) = 0$ and $y'(1) = 0$. Let $f(x) = (x-1)^2 - 1$.

   (a) Show that $f(x)$ satisfies the boundary conditions. Find constants a, b, and c so that $y_p(x) = ax^2 + bx + c$ satisfies the differential equation and the boundary condition at $x = 1$ (it doesn't have to satisfy $y_p(0) = 0$).

(b) Find constants C_1 and C_2 so that

$$y(x) = C_1 \cos \frac{x}{\sqrt{\epsilon}} + C_2 \sin \frac{x}{\sqrt{\epsilon}} + y_p(x)$$

is the exact solution of the BVP, where $y_p(x)$ is your result from part (a). Note that C_1 and C_2 may depend on ϵ, and your solution should satisfy the differential equation and both boundary conditions.

(c) Apply the centered difference approximation of the second derivative $D_+D_-y = \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1})$ to the differential equation $\epsilon y'' + y = (x-1)^2 - 1$. Write the equation satisfied by the ith point, where i is the index of a point in the interior of the domain, away from both boundaries.

(d) Write the equations that must be satisfied by the points near the right-hand boundary in two ways: 1) using the first-order approximation D_- and 2) using the second order approximation D_2 derived in homework 2 (check the case $i = n$). How do these methods affect the structure of the matrix associated with this problem? Recall: $D_2 f(x) = \frac{1}{2h}(3f(x) - 4f(x-h) + f(x-2h))$.

(e) You now have 2 methods for solving this problem; one using D_- for the $x = 1$ boundary condition and one using D_2. Write the system $Aw = r$ of $n+2$ equations you have derived in matrix form for each case, where A is the $(n+2) \times (n+2)$ matrix whose ith row gives the equation for the ith grid point $x_i = (i-1)h$, $i = 0, \ldots, i+1$.

Are both systems tridiagonal?

Let $\epsilon = 1/64$ and compute solutions for $N = 2^k$ subdivisions where $k = 5, \ldots, 12$ using the D_- approximation for the $x = 1$ boundary condition using the tridiagonal solver discussed in class.

You **don't** have to solve the system that uses the D_2 approximation for the $x = 1$ boundary.

(f) Compute the absolute error in both the l_2 and l_∞ norms and plot the error convergence vs. step size h for each method. Discuss the apparent effect of the order of accuracy of the approximation used at $x = 1$ on the computed solution over the whole domain.

3. page 672, problem 15a (deflection of a beam). The deflection of a beam's centerline, $u(x)$, satisfies the equation $u'' - \frac{T}{EI}u = -\frac{w}{2EI}x(L-x)$, subject to the boundary conditions $u(0) = u(L) = 0$. The textbook asks you to compute the beam deflection at 1-inch intervals, which you should do. Also calculate two additional cases: 2-inch and 4-inch intervals. Plot all three cases on the same axes (use difference symbols to distinguish the cases). Give the maximum beam deflection computed in each case.