Math 471-001 Fall 2013 Computing Project 2 due : Thursday, Nov. 21

2D Boundary value problems

1. **Temperature.** A metal plate is described by the unit square, $D = \{(x, y) : 0 \le x, y \le 1\}$. The temperature on the plate is given by the function $\phi(x, y)$, and ϕ satisfies the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ on D with boundary conditions $\phi(x, 1) = 1$, $\phi(x, 0) = \phi(0, y) = \phi(1, y) = 0$. Physically this means that there are no internal heat sources and that one edge of the plate is heated to a constant temperature while the other edges are held at a lower temperature.

   We solve for the temperature on the plate using the finite difference scheme $(D^x_+ D^x_- + D^y_+ D^y_-)w_{ij} = 0$ with mesh size $h = 1/(n + 1)$. This yields a linear system denoted by $A_h w_h = f_h$ where $w_h = \{w_{ij}\}$ is the numerical solution vector with components $w_{ij} \approx \phi(x_i, y_j)$ and the mesh points are given by (x_i, y_j) with $x_i = ih$, $y_j = jh$ for $i, j = 0, 1, 2, \ldots, n + 1$.

   (a) Use Matlab to solve the linear system by Jacobi's method with mesh sizes $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$. Use $w^{(0)} = 0$ for the starting vector and take $\dfrac{\|r_k\|_\infty}{\|r_0\|_\infty} \le 10^{-2}$ as the stopping criterion, where $r_k = A_h w_h^{(k)} - f_h$. To keep your code simple, the numerical solution $w_h = \{w_{ij}\}$ should be coded as a matrix of dimension $(n + 2) \times (n + 2)$ containing the unknown interior temperature values and the known boundary values.
   Do not form the full matrix A_h (because it's sparse and this would waste memory); instead use the component form of Jacobi's method which has only five nonzero entries for each equation. Present your results as described below, and include a copy of your code.

   (b) For each value of h, plot the computed temperature w_{ij} at the final step using a contour plot and a mesh plot (to save printer ink do not use `surf`). Type `help contour` and `help mesh` in Matlab for instructions. Use the `subplot` command to get several graphs on one plot, as in the lecture notes.

   (c) For each value of h report the number of iterations required to reach the stopping criterion. Summarize your results with a brief writeup.

2. **Wind-driven ocean circulation.** A model of ocean circulation in a rectangular basin $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le L\}$ is defined by

$$\nabla^2\psi + \alpha\frac{\partial\psi}{\partial x} = A\sin\frac{\pi y}{L}, \qquad \psi(0, y) = \psi(1, y) = \psi(x, 0) = \psi(x, L) = 0,$$

   where ψ is the stream function of the flow. The Laplacian of the stream function $\nabla^2\psi$ is the vorticity. The constant $\alpha > 0$ represents the effects of the Earth's rotation; $\alpha = 0$ corresponds to a non-rotating planet. The constant A on the right-hand side models the long-term mean forcing of the north Atlantic ocean due to wind stress; storms typically travel west to east in the midlatitudes. The constant L represents the aspect ratio of the basin's north-south to east-west sides.

   Set the constants $\alpha = 20$, $L = 2/3$, and $A = 25$.

In 1948 H. Stommel [2] used this model to explain why western boundary currents (i.e., the Gulf Stream, Kuroshio, and Brazil currents) are much stronger than eastern boundary currents (i.e., the California, Canary, and Benguela currents). A derivation of the model and its exact solution is provided in [1, chapter 9] (and Math 454).

(a) Use the mesh (x_i, y_j), where $x_i = ih$, $i = 0, 1, 2, \ldots, n+1$ with $h = 1/(n+1)$ and $y_j = jk$, for $j = 0, 1, 2, \ldots, m+1$ with $k = L/(m+1)$. Assume that n and m are chosen so that $k = h$ (use equal spacing in both directions). Apply the 5-point stencil for the Laplace operator and the centered finite difference operator $D_0^x = \frac{1}{2h}(\psi(x+h, y) - \psi(x-h, y))$ to the model to derive a finite difference scheme.

(b) Although the physical domain is rectangular, the finite difference scheme can still be written as a square matrix, $A_h w_h = f_h$, where A_h is $nm \times nm$. Choose $n = 5$ and $m = 3$ and apply column ordering to write the matrix A_h. Row I of the matrix corresponds to the mesh point ij according to $I(i, j) = (i-1)m + j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Note that the A_h matrix can be written in the form

$$A_h = \frac{1}{h^2} \begin{pmatrix} T & D_1 & & & \\ D_2 & T & D_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & D_1 \\ & & & D_2 & T \end{pmatrix}$$

where each T is a tridiagonal matrix and $D_{1,2}$ are diagonal matrices. Define T and $D_{1,2}$ and determine their sizes in terms of n and m.

Note that since $D_1 + D_2 = I$, the matrix A_h is strictly diagonally dominant; hence we can apply iterative methods such as Jacobi, Gauss-Seidel, and SOR to this problem.

(c) Jacobi's method for this problem can be written as

$$w_{ij} = \frac{1}{4}\left[\left(1 - \frac{\alpha h}{2}\right) w_{i-1,j} + \left(1 + \frac{\alpha h}{2}\right) w_{i+1,j} + w_{i,j-1} + w_{i,j+1} - Ah^2 \sin\frac{\pi y_j}{L}\right].$$

Write a Matlab program that solves the problem using Gauss-Seidel with initial guess $w^{(0)} = 0$. Use the same stopping criterion as the previous problem, $\frac{\|r_k\|_\infty}{\|r_0\|_\infty} \le 10^{-2}$, on meshes with $(n, m) \in \{(14, 9), (29, 19), (59, 39), (119, 79)\}$. As before, to keep the code simple, the numerical solutions $w_{ij} \approx \psi(x_i, y_j)$ should be stored in a matrix of dimension $(n+2) \times (m+2)$. Plot the solutions using `subplot` and `contour`, and report the number iterations required to converge in each case.

(d) Repeat part (c) using SOR with $\omega = 1.75, 1.5, 1.25$ (you do not need to reproduce the contour plots for each value of ω – just verify that your SOR code produces the same solution as your Gauss-Seidel code and report the number of iterations required in each case). Based on your results, how does the best value of ω appear to vary with the mesh size? Does this result make sense, given what you know about the eigenvalues of finite difference matrices and Young's theorem?

References

[1] R. Malek-Madani. *Physical Oceanography: A Mathematical Introduction with Matlab*. CRC Press, 2012.

[2] H. Stommel. "The western intensification of wind-driven ocean currents," *Transactions of the American Geophysical Union*, 1948. 29:202-206.