Math 471-001 Fall 2013 Review sheet for midterm exam

The midterm exam will be in class on Thursday, October 31. You may use one sheet of notes (one side of one page, 8.5 in × 11 in). You may use a non-programmable calculator to do arithmetic, but to receive full credit you must show intermediate steps. The exam will cover up to and including the inverse power method (≈ Thursday, October 24 class).

1. **True or False?** Give a reason to justify your answer.
   (a) $(10101.01)_2 = (21.25)_{10}$.
   (b) If two numbers with n significant digits are subtracted, the result also has n significant digits.
   (c) When the derivative $f'(x)$ is approximated by the forward difference approximation $D_+f(x)$ with step size h, the roundoff error dominates the truncation error for large h.
   (d) $D_+D_+f(x) = f''(x) + O(h)$
   (e) $D_+D_-f(x) = D_-D_+f(x)$
   (f) The central difference approximation defined by $D_0f(x) = \frac{1}{2h}(f(x+h)-f(x-h))$ is a 2nd-order accurate approximation of the derivative $f'(x)$.
   (g) The approximation $f'(x) \approx D_0f(x)$ is exact if f is a polynomial of degree less than or equal to three.
   (h) The secant method converges faster than Newton's method.
   (i) In solving an $n \times n$ system of linear equations by Gaussian elimination, if n is increased by a factor of 10, then the operation count is increased by a factor of approximately 10^3.
   (j) In solving an $n \times n$ system of linear equations by Gaussian elimination, partial pivoting is recommended even if the pivots are nonzero in order to reduce the operation count.
   (k) If A is invertible, then $\kappa(A) \geq 1$.
   (l) If x is the input to a system and Ax is the output, then the norm of the output is bounded by $\|A\|$ times the norm of the input.
   (m) Gaussian elimination is an unstable method for solving $Ax = b$ because it can replace a an ill-conditioned matrix A by a well-conditioned matrix U.
   (n) In solving a linear system $Ax = b$ by a numerical method, if the residual is small then the error is also guaranteed to be small.
   (o) In solving a two-point boundary value problem using a 2nd order finite difference scheme, if the mesh size h is reduced by one-half, then the norm of the error is also reduced by one-half.
   (p) Consider an iterative method $x_{k+1} = Bx_k + c$ for solving $Ax = b$. If $\|B\| < 1$, then the method converges for all initial vectors x_0.
   (q) In solving the linear system $Ax = b$ where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ by iterative methods, one step of the Gauss-Seidel method reduces the norm of the error as much as two steps of Jacobi's method.

(r) If $\lambda = 0$ is an eigenvalue of A, then A is invertible.

(s) Wilkinson's example shows that the coefficients of a polynomial can depend sensitively on the roots.

(t) When the power method is applied to find the largest eigenvalue and the corresponding eigenvector of a matrix, the vectors are normalized at each step in order to increase the rate of convergence.

2. Let $f(x) = \sqrt{1+x^2} - 1$ and $g(x) = x^2/(\sqrt{1+x^2}+1)$. Show that $f(x) = g(x)$ for all x. Which expression is better to use when x is a small number? Explain.

3. Consider a finite difference approximation $f'(x) \approx \frac{af(x+h)+bf(x)+cf(x-h)}{h}$, where a, b and c are unknown coefficients. The forward approximation D_+f has $(a,b,c) = (1,-1,0)$ and is first-order accurate. The central approximation $D_0 f$ has $(a,b,c) = (\frac{1}{2}, 0, -\frac{1}{2})$ and is second order accurate. Are there any values of (a,b,c) that yield 3rd order accuracy?

4. Suppose the equation $f(x) = x^2 - 5 = 0$ is solved by the bisection method with $a = 0$ and $b = 3$. How many steps are needed to ensure the error is less than 10^{-3}?

5. Below is an algorithm for the bisection method. Find and correct any errors.

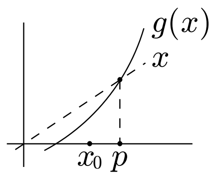
Algorithm 1: Bisection method

Input : $f(x), a, b$
Output: $x \approx p$ where $f(p) = 0$
Assume $f(a) \cdot f(b) > 0$.
$n = 0$, $a_0 = a$, $b_0 = b$
for $n = 0 : n_{\max}$ **do**
    $x_n = \frac{a_n - b_n}{2}$ `% current estimate of root`
    if $f(x_n) \cdot f(a_n) < 0$ **then**
        $a_{n+1} = x_n$
        $b_{n+1} = b_n$
    else
        $a_{n+1} = a_n$
        $b_{n+1} = x_n$
    end
end

6. Consider solving $f(x) = 0$. (a) State one advantage of Newton's method over the bisection method. (b) State one advantage of the bisection method over Newton's method.

7. Let $g(x) = x^2 - \frac{1}{2}x + \frac{1}{2}$. Find the fixed points of $g(x)$. Suppose fixed-point iteration $x_{n+1} = g(x_n)$ is applied with $x_0 = 1$. Which fixed point does the iteration converge to?

8. Consider fixed-point iteration $x_{n+1} = g(x_n)$. The plot shows a function $y = g(x)$, the line $y = x$, the fixed point p, and the initial guess x_0. Does the sequence x_n converge to p? Explain.



9. Solve $Ax = b$ by Gaussian elimination with partial pivoting.

a) $A = \begin{pmatrix} 0 & 4 & -15 \\ 10 & 0 & 15 \\ 1 & -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} -12 \\ 100 \\ 0 \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

10. Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find a vector x such that $\|Ax\|_\infty = \|A\|_\infty$.

11. Consider the linear system $2x_1 - x_2 = 1,\ -x_1 + 2x_2 - x_3 = 0, -x_2 + 2x_3 - x_4 = 0, -x_3 + 2x_4 = 1$. Write the system in matrix form $Ax = b$ and solve for x by LU factorization.

12. Find the e-values and e-vectors of the following matrices. Do this by hand (but you may check your answers using Matlab - type `help eig` to learn the command.)

a) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

13. Consider the linear system $2x_1 - x_2 = 1, -x_1 + 2x_2 - x_3 = 0, -x_2 + 2x_3 = 1$, with solution $x_1 = x_2 = x_3 = 1$. a) Write out Jacobi's method in component form. Take one step starting from the zero vector. Compute the error norms $||e_0||_\infty, ||e_1||_\infty$. b) Repeat for Gauss-Seidel.

14. Let $A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

a) For which of these does Jacobi's method converge?

b) For which of these does Gauss-Seidel converge?

15. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find $\max_{x \neq 0} R_A(x)$.

16. Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Take two steps of the power method starting from $v^{(0)} = \frac{1}{\sqrt{3}}(1, 1, 1)^T$ (i.e., compute $v^{(k)}$ and $R_A(v^{(k)})$ for $k = 1, 2$).