**Math 471-001 Fall 2013 Homework 2 due : Thursday, Sept. 26**

## finite differences

1. The forward and backward difference approximation are defined by

$$D_+f(x) = \frac{f(x+h)-f(x)}{h}, \qquad D_-f(x) = \frac{f(x)-f(x-h)}{h}.$$

   (a) Show that $D_+D_-f(x) = \dfrac{f(x+h)-2f(x)+f(x-h)}{h^2}$.

   (b) Show that $D_+D_-f(x) = f''(x) + O(h^2)$ and find the asymptotic error constant.

2. In this problem we derive a second order accurate one-sided approximation to $f'(x)$ using $f(x)$, $f(x-h)$, and $f(x-2h)$ using the method of undetermined coefficients. Assume

$$D_2f(x) = af(x) + bf(x-h) + cf(x-2h)$$

   and solve for the coefficients $a$, $b$, and $c$, so that $D_2f(x) = f'(x) + O(h^2)$. Define the operator $D_2f(x)$ and the leading term of its truncation error in your answer.

## rootfinding

3. Consider $f(x) = x^2 - 7$. It follows that $f$ has a root at $x = p$ where $p \in [2,3]$. Compute an approximation to $p$ by the following methods. Use Matlab and print the answers to 15 digits.

   (a) bisection method, starting interval $[a,b] = [2,3]$

   (b) fixed-point iteration with $g_1(x) = 7/x$ and $g_2(x) = x - f(x)/3$, starting value $x_0 = 2.5$

   (c) Newton's method, starting value $x_0 = 2.5$

   Take 6 steps in each case. Present the results in a table with columns as below for each method. Do the results agree with the theory discussed in class?

   column 1 : $n$ (step)

   column 2 : $x_n$ (approximation)

   column 3 : $f(x_n)$ (residual)

   column 4 : $|p - x_n|$ (error)

4. In class we discussed the example, "Volume of Chlorine Gas" on page 102. This example uses Newton's method to compute the volume of a gas given by van der Waal's equation of state, where the initial guess $V_0$ is given by the ideal gas law. We saw that $V_0$ has 2 correct digits and $V_1$ has 5 correct digits. How many correct digits does $V_2$ have?

5. The screened Coulomb potential is given by $\phi(r) = \frac{e^{-\kappa r}}{r}$, where $r$ is the distance from a charged particle to a point in space and $\kappa$ controls the screening effect. Let $\kappa = \frac{1}{2}$. Use Newton's method to find the value of $r$ for which $\phi(r) = 0.005$. Let $r_0 = 1$ be the starting value and take two steps. How many digits in the final result can you trust?

6. The equation

$$r\left(1-\frac{u}{q}\right)=\frac{u}{1+u^2},$$

arises from a dynamic population model of spruce budworms, which can defoliate balsam fir forests [1]. The constant parameter $r$ measures the reproduction rate of the budworm and the constant $q$ is a measure of the available resources in the environment to support the budworm population. The population variable is $u$. Nonzero solutions of this equation correspond to steady states of a budworm population.

   (a) Suppose that environmental observations show that $r = 0.35$ and $q = 20$. How many possible steady states are there? (Hint: plot $f(u) = r\left(1-\frac{u}{q}\right)$ and $g(u) = \frac{u}{1+u^2}$ on the same set of axes.)

   (b) Use Matlab and the method of your choice to find the minimum steady state (known as the refuge population) and the maximum steady state (the outbreak population). Report your initial guess (or initial interval), how you determined the stopping criterion, estimate how accurate your results are, and report the number of iterations it took to achieve your desired accuracy.

   (c) Suppose that the forestry service is able to spray the balsam fir trees with an agent that discourages budworms' appetites to a point where $q = 8$. How many possible steady states are there? Comment on the effectiveness of this strategy.

   **Note:** The number of steady states changes depending on the parameter values $r$ and $q$; in the study of dynamical systems, this is known as a *hysteresis effect*. These phenomena occur in many models across science, and are studied in detail in Math 404 – Intermediate Differential Equations.

7. Consider the following system of nonlinear equations.

$$f(x,y)=(x-1)^2+y^2-4=0, \quad g(x,y)=xy-1=0$$

   Finding the points $(x, y)$ that satisfy both equations corresponds to finding the intersections of a circle and a hyperbola. Find an approximate solution using Newton's method for systems. Take two steps starting from $(x_0, y_0) = (3, 0)$. Present the iterates $(x_i, y_i)$ and residual values $f(x_i, y_i), g(x_i, y_i)$ for $i = 0, 1, 2$.

**linear algebra**

8. page 148 : problems 4a, b; 7a (warmup exercises on matrices)

9. page 148 : problem 10 (warmup exercises on linear algebra)

10. page 157 : problem 1 (Gaussian elimination)

## References

[1] Murray, J.D. *Mathematical Biology,* 3rd edition. Springer, 2002.