Math 471-001 Fall 2013 Homework 4 due : Tuesday, Oct. 29

iterative methods

1. Consider the linear system $2x_1 + x_2 = 1, \quad x_1 + 2x_2 = -1$.

   (a) Write the system in matrix form and solve it by LU factorization.

   (b) Write out Jacobi's method in component form and take three steps starting from the initial guess $x_0 = [0,0]^T$. Present the results in a table with the following format:

   column 1: k (iteration step)

   column 2: $x_1^{(k)}$ (1st component of computed solution vector at step k).

   column 3: $x_2^{(k)}$ (2nd component of computed solution vector at step k).

   column 4: $\left\|e^{(k)}\right\|_\infty$ (error norm at step k).

   column 5: $\left\|e^{(k)}\right\|\infty / \left\|e^{(k-1)}\right\|_\infty$ (ratio of successive error norms at step k).

   Find the iteration matrix B_J and compute $\|B_J\|_\infty$ and $\rho(B_J)$. Does the method converge?

   (c) Repeat part (b) using Gauss-Seidel.

   (d) Repeat part (b) using optimal SOR.

2. Consider $Ax = b$, where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   (a) In class we showed that the error in the Gauss-Seidel method is given by

   $$e^{(k)} = \left(\frac{1}{4}\right)^k \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

   for $k \geq 1$, using the eigenvalues and eigenvectors for the iteration matrix B_{GS}. Following the same procedure, derive an analogous expression for Jacobi's method. Assume the starting guess is the zero vector.

   (b) Use Matlab to plot $\rho(B_\omega)$, the spectral radius of the SOR iteration matrix for $0 \leq \omega \leq 2$. Use the commands `eig`, `abs`, and `max` to produce the plot. Make sure to use a fine enough mesh in the variable ω to resolve the details of the function (1000 points on the interval $\omega \in [0,2]$ is sufficient). This plot is typical for matrices appearing in Young's theorem. Suppose we don't know the exact value of the optimal SOR parameter – in using an approximate iterative method, is it better to overestimate or underestimate ω^*? Explain your reasoning.

3. Consider the iteration method $x^{(k+1)} = Bx^{(k)} + c$ and assume that $\|B\| = \alpha < 1$. We know that the error satisfies $\left\|x - x^{(k+1)}\right\| \leq \alpha \left\|x - x^{(k)}\right\|$, which is an important theoretical bound, but the right hand side cannot be computed in practice because although we know $x^{(k)}$, we don't know x. Here we derive an alternative error bound that can be computed in practice.

   (a) Show that $I - B$ is invertible and that $\|(I-B)^{-1}\| \leq \frac{1}{1-\alpha}$.

(b) Show that $\left\| x - x^{(k+1)} \right\| \le \frac{\alpha}{1-\alpha} \left\| x^{(k+1)} - x^{(k)} \right\|$.

Note: the bound in part (b) can be computed because we know both $x^{(k+1)}$ and $x^{(k)}$.

special matrices

4. Determine whether or not the following matrices are positive definite. If not, give an example $x \neq 0$ such that $x^T A x \le 0$.

a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

5. page 220, problems 2 and 3

eigenvalues

6. Find the eigenvalues and eigenvectors of the following matrices. Do this by hand, but you may use Matlab to check your work.

a) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ e) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$