Math 471-001 Fall 2013 Homework 3 due : Thursday, Oct. 10

linear algebra

1. Which of the following matrices are invertible? Justify your answer. For those that are singular, find a nonzero vector x such that $Ax = 0$.

   (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 3 & 3 \end{pmatrix}$

2. page 159, problem 13 (this one should look familiar). Solve using Gaussian elimination and partial pivoting.

3. page 169, problem 14, solve the system in three ways:

   (a) Gaussian elimination with no pivoting, 3 decimal digit arithmetic with rounding.

   (b) Gaussian elimination with partial pivoting, 3 decimal digit arithmetic with rounding.

   (c) Matlab backslash command.

matrix and vector norms

4. page 180, problems 1 and 2b

5. The unit circle is defined as the points $\boldsymbol{x} = [x, y]^T$ that satisfy $\|\boldsymbol{x}\| = 1$. Recall that

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|\,, \qquad \|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \qquad \|\boldsymbol{x}\|_\infty = \max_{1 \le i \le n} |x_i|\,.$$

   (a) Use Matlab to plot the unit circle of all three norms; that is, plot the sets S_p, where

$$S_p = \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\|_p = 1\}, \qquad p \in \{1, 2, \infty\}.$$

   Hint: use `axis equal` to plot the x and y axes with the same scale.

   (b) Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Find $\|A\|_2$ and $\|A\|_\infty$.

   (c) Find unit vectors $\boldsymbol{x}_1$, and $\boldsymbol{x}_2$, such that

$$\begin{matrix} \|\boldsymbol{x}_1\|_2 = 1 \\ \|A\boldsymbol{x}_1\|_2 = \|A\|_2 \end{matrix}, \qquad \begin{matrix} \|\boldsymbol{x}_2\|_\infty = 1 \\ \|A\boldsymbol{x}_2\|_\infty = \|A\|_\infty \end{matrix}.$$

   Find the exact solutions for $\boldsymbol{x}_2$. Your answer for $\boldsymbol{x}_1$ does not have to be exact; an approximation with error less than 1×10^{-6} is acceptable (note that while it is possible to find the exact answer, it may be significantly faster to find an accurate approximation).

error analysis

6. Let $A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix},\ x = \begin{pmatrix} 2 \\ -2 \end{pmatrix},\ x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

   (a) Show that x is the exact solution of $Ax = b$.

   (b) Think of x_1 and x_2 as approximations to x. Compute the errors e_1, e_2, and residuals r_1, r_2 corresponding to x_1, x_2.

   (c) Find $\|A\|_\infty$, $\|A^{-1}\|_\infty$, and $\kappa_\infty(A)$.

   (d) In class we proved the following theorem relating the relative error, relative residual, and condition number:

$$\frac{\|E\|}{\|x\|} \leq \kappa(A) \cdot \frac{\|r\|}{\|b\|}.$$

   Show that this result holds for the approximate solutions x_1 and x_2 given above.

7. Derive the following result, which was stated in class.

$$\left.\begin{array}{l} Ax = b \\ \tilde{A}\tilde{x} = b \end{array}\right\} \Rightarrow \frac{\|x - \tilde{x}\|}{\|\tilde{x}\|} \leq \kappa(A) \frac{\left\|A - \tilde{A}\right\|}{\|A\|}$$

   Note: This results says that in solving a linear system $Ax = b$, the condition number of the matrix controls the relative error in the solution due to perturbations in the matrix.

LU factorization

8. This exercise concerns the matrix form of partial pivoting. Let A be a 3×3 matrix and suppose we apply LU factorization to obtain $M_2 P_2 M_1 P_1 A = U$, where U is upper triangular and

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   (a) Compute $\tilde{M}_1 = P_2 M_1 P_2$.

   (b) Show that $P_2 M_1 = \tilde{M}_1 P_2$. Note that this implies that $M_2 \tilde{M}_1 P_2 P_1 A = U$.

   (c) Compute $P = P_2 P_1$ and $L = \tilde{M}_1^{-1} M_2^{-1}$.

   (d) Show that $PA = LU$.

   (e) Let $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$. Find P_1, M_1, P_2, and M_2 by carrying out Gaussian elimination with partial pivoting. Note that P_1, P_2 here are not necessarily the same as above. Also note that if no pivoting is required at a certain step k, then the permutation matrix P_k at that step is taken to be the identity matrix I. Write down the resulting P, L, U for the given A and show that $PA = LU$. Noting that $Ax = b \Rightarrow PAx = Pb \Rightarrow LUx = Pb$, compute Pb and solve for x by applying forward and backward substitution.

computing project 1 : due Tuesday, October 22

9. Complete computing projects A and B.