linear algebra

1. Which of the following matrices are invertible? Justify your answer. For those that are singular, find a nonzero vector $x$ such that $Ax = 0$.

   (a) \[
   \begin{pmatrix}
   0 & 1 \\
   1 & 0 \\
   \end{pmatrix}
   \]
   (b) \[
   \begin{pmatrix}
   1 & -1 \\
   -1 & 1 \\
   \end{pmatrix}
   \]
   (c) \[
   \begin{pmatrix}
   1 & -1 \\
   1 & 1 \\
   \end{pmatrix}
   \]
   (d) \[
   \begin{pmatrix}
   1 & 0 & 2 \\
   -1 & 3 & 1 \\
   0 & 3 & 3 \\
   \end{pmatrix}
   \]

2. page 159, problem 13 (this one should look familiar). Solve using Gaussian elimination and partial pivoting.

3. page 169, problem 14, solve the system in three ways:

   (a) Gaussian elimination with no pivoting, 3 decimal digit arithmetic with rounding.
   (b) Gaussian elimination with partial pivoting, 3 decimal digit arithmetic with rounding.
   (c) Matlab backslash command.

matrix and vector norms

4. page 180, problems 1 and 2b

5. The unit circle is defined as the points $x = [x, y]^T$ that satisfy $\|x\| = 1$. Recall that

   \[
   \|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}, \quad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.
   \]

   (a) Use Matlab to plot the unit circle of all three norms; that is, plot the sets $S_p$, where $S_p = \{x \in \mathbb{R}^2 : \|x\|_p = 1\}$, $p \in \{1, 2, \infty\}$.

   Hint: use `axis equal` to plot the $x$ and $y$ axes with the same scale.

   (b) Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Find $\|A\|_2$ and $\|A\|_\infty$.

   (c) Find unit vectors $x_1$, and $x_2$, such that

   $\|x_1\|_2 = 1$, $\|x_2\|_\infty = 1$, $\|Ax_1\|_2 = \|A\|_2$, $\|Ax_2\|_\infty = \|A\|_\infty$.

   Find the exact solutions for $x_2$. Your answer for $x_1$ does not have to be exact; an approximation with error less than $1 \times 10^{-6}$ is acceptable (note that while it is possible to find the exact answer, it may be significantly faster to find an accurate approximation).

error analysis
6. Let \( A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix} \), \( b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix} \), \( x = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \), \( x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), \( x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

(a) Show that \( x \) is the exact solution of \( Ax = b \).
(b) Think of \( x_1 \) and \( x_2 \) as approximations to \( x \). Compute the errors \( e_1, e_2 \), and residuals \( r_1, r_2 \) corresponding to \( x_1, x_2 \).
(c) Find \( \| A \|_\infty, \| A^{-1} \|_\infty \), and \( \kappa_\infty(A) \).
(d) In class we proved the following theorem relating the relative error, relative residual, and condition number:
\[
\frac{\| E \|}{\| x \|} \leq \kappa(A) \cdot \frac{\| r \|}{\| b \|}.
\]
Show that this result holds for the approximate solutions \( x_1 \) and \( x_2 \) given above.

7. Derive the following result, which was stated in class.
\[
Ax = b \quad \Rightarrow \quad \frac{\| x - \tilde{x} \|}{\| \tilde{x} \|} \leq \kappa(A) \frac{\| A - \tilde{A} \|}{\| A \|}.
\]
Note: This results says that in solving a linear system \( Ax = b \), the condition number of the matrix controls the relative error in the solution due to perturbations in the matrix.

**LU factorization**

8. This exercise concerns the matrix form of partial pivoting. Let \( A \) be a \( 3 \times 3 \) matrix and suppose we apply \( LU \) factorization to obtain \( M_2P_2M_1P_1A = U \), where \( U \) is upper triangular and
\[
M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

(a) Compute \( \tilde{M}_1 = P_2M_1P_2 \).
(b) Show that \( P_2M_1 = \tilde{M}_1P_2 \). Note that this implies that \( M_2\tilde{M}_1P_2P_1A = U \).
(c) Compute \( P = P_2P_1 \) and \( L = \tilde{M}_1^{-1}M_2^{-1} \).
(d) Show that \( PA = LU \).
(e) Let \( A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \). Find \( P_1, M_1, P_2, \) and \( M_2 \) by carrying out Gaussian elimination with partial pivoting. Note that \( P_1, P_2 \) are not necessarily the same as above. Also note that if no pivoting is required at a certain step \( k \), then the permutation matrix \( P_k \) at that step is taken to be the identity matrix \( I \). Write down the resulting \( P, L, U \) for the given \( A \) and show that \( PA = LU \). Noting that \( Ax = b \Rightarrow PAx = Pb \Rightarrow LUx = Pb \), compute \( Pb \) and solve for \( x \) by applying forward and backward substitution.

**Computing project 1** : due Tuesday, October 22

9. Complete computing projects A and B.