Math 471-001 Fall 2013 Homework 5 due : Thursday, Nov. 14

eigenvalues

1. Suppose λ_i is an eigenvalue of an invertible matrix A, with eigenvector v_i. Show that $1/\lambda_i$ is an eigenvalue of A^{-1}. Note that this implies that for each i, $i = 1, \ldots, n$, $1/\lambda_i$ is an eigenvalue of A^{-1} with eigenvector v_i, which we used in class to define the inverse power method for computing eigenvalues.

2. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and recall that in class we computed the eigenvalues $\lambda_{1,2}$ and corresponding orthonormal eigenvectors $q_{1,2}$ of A. Now consider $Ax = b$, where $b = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. Let $Q = [q_1 \;\; q_2]$ be the orthogonal matrix of eigenvectors of A. Show that $Ax = b$ can be written as $\Lambda Q^T x = Q^T b$, where Λ is the diagonal matrix of eigenvalues of A.

   Compute $x = \lambda_1^{-1}(q_1^T b)q_1 + \lambda_2^{-1}(q_2^T b)q_2$ by explicitly evaluating the expression on the right-hand side and verifying that it is a solution of the linear system. This illustrates the <u>spectral method</u>, a method that expresses the solution of $Ax = b$ in terms of the eigenvalues and eigenvectors of A.

3. In class we considered the e-value problem for the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$. The largest eigenvalue is $\lambda_1 = 5.214319743377535$, as can be confirmed using the Matlab `eig` command. A table of results was presented for the power method, shifted inverse iteration with $\mu = 5$, and the Rayleigh quotient iteration. The initial guess for the eigenvector was $v^{(0)} = \frac{1}{\sqrt{3}}[1, 1, 1]^T$. The computed e-values were shown for iterations $k = 0, 1, 2$. Repeat the computations to extend the table to $k = 4$. Use Matlab to present 15 decimal digits for the computed e-values (type `format long`). For each method underline the correct digits at steps $k = 3$ and $k = 4$.

4. Gerschgorin circle theorem.

   (a) Use the Gerschgorin circle theorem to estimate the eigenvalues $A = \begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}$.

   Set $\epsilon = 0.1$ and change the code presented in the lecture notes to plot the Gerschgorin circles of A.

   (b) Prove that an $n \times n$ matrix A that is strictly diagonally dominant is nonsingular.

interpolation

5. Find the Taylor series for $f(x) = \sin x$ about $x = 0$. Using Matlab, plot the Taylor polynomials $p_n(x)$ of degree $n = 1, 3, 5, 7$ on the same plot on the interval $x \in [-4\pi, 4\pi]$. Also plot the original function $f(x)$, and label each curve. Viewing the Taylor polynomials as approximations of the original function, describe in words what happens to the approximation as the degree of the polynomial increases.

6. page 350, problem 4a,b,c

7. page 352, problem 14

8. In class we saw that the Taylor series for $f(x) = \frac{1}{1+25x^2}$ about $x = 0$ converges for $|x| < \frac{1}{5}$. Now we consider the expansion about $x = \frac{1}{5}$.

   (a) Let $T_i(x)$ be the Taylor polynomial of degree i for $f(x)$ about $x = \frac{1}{5}$. Using Matlab, plot $f(x)$ and $T_i(x)$ for $i = 0, 1, 2, 3$ on the interval $-1 \le x \le 1$. How do the results compare with the results given in class?

   (b) On what interval does the Taylor series about $x = \frac{1}{5}$ converge? You may answer on the basis of the numerical results from part (a), but for full credit you should give a mathematical justification of your answer.

9. Let $f(x) = 1/x$ and take $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$.

   (a) Find the Lagrange form, Newton form, and standard form of the interpolating polynomial $p_3(x)$. Check your answer in each case by verifying that $p_3(x)$ interpolates $f(x)$ at the given points.

   (b) Find an upper bound for the maximum error $e = \max_{1 \le x \le 4} |f(x) - p_3(x)|$.

10. Let x_0, x_1, and x_2 be 3 distinct points and define $p(x) = L_0(x) + L_1(x) + L_2(x)$. Find the standard form of $p(x)$. Solve this problem in two ways: 1) by direct computation using the definition of the Lagrange polynomials, 2) by applying the theorem which says there is a unique polynomial of degree $\le n$ which interpolates a function at $n + 1$ points (hint: what function does $p(x)$ interpolate?).

11. Let $f(x) = \sqrt{1 - x^2}$ for $x \in [-1, 1]$. Using Matlab, plot the following approximations and comment on the results.

    (a) Taylor polynomials of degree $n = 0, 2, 4, 6$ about $x = 0$.

    (b) piecewise linear interpolant on a uniform mesh with mesh spacing $h = 2/n$ for $n = 2, 4, 8, 16$.

    (c) piecewise linear interpolant using Chebyshev points with $\Delta\theta = \pi/n$ for $n = 2, 4, 8, 16$.