Name: ____________________________ UM ID #: _________________________

Instructions and notes

1. Turn off and put away your cell phone and other electronic devices. Please ensure that they will be silent for the duration of the exam.

2. You may use one page of notes (one side of one piece of letter size paper, 8.5 in × 11 in).

3. The use of calculators or any other electronic aids are not permitted on this exam.

4. Write your solutions clearly; no credit will be given for illegible solutions. Please clearly mark your ultimate result for each problem (i.e., by drawing a box or a circle around your answer).

5. This exam has 5 problems spread across 6 double-sided pages.

6. If you need more room to write than the space provided by the exam, write your name on each additional sheet of paper you turn in.

Academic integrity statement

Please copy the following statement, and sign below your writing: “I pledge that I have neither given nor received any unauthorized assistance regarding this exam.”

signature: ____________________________
possibly useful formulae:

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad \text{(Cartesian)} \]

\[ = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2}, \quad \text{(spherical)} \]

\[ \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases} \]

\[ \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n = 0 \\ \frac{L}{2} & m = n \neq 0 \end{cases} \]

\[
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B, \quad \cos A \cos B = \frac{1}{2} \left( \cos(A - B) + \cos(A + B) \right) \\
\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B, \quad \sin A \sin B = \frac{1}{2} \left( \cos(A - B) - \cos(A + B) \right) \\
\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B, \quad \cosh A \cosh B = \frac{1}{2} \left( \cosh(A + B) + \cosh(A - B) \right) \\
\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B, \quad \sinh A \sinh B = \frac{1}{2} \left( \cosh(A + B) - \cosh(A - B) \right)
\]

heat equation:

\[ K_0 = \text{thermal conductivity}, \quad c = \text{specific heat}, \quad \rho = \text{mass density} \]

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{(1-d)}, \quad \frac{\partial u}{\partial t} = k \nabla^2 u \quad \text{(2-d & 3-d)}, \quad k = \frac{K_0}{c \rho} = \text{const.} \]

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<thead>
<tr>
<th>Problem</th>
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1. **True / False.** Give a reason to justify your answer.
   (10 points : 5 parts, 2 points each)

   (a) It is possible to find coefficients $b_n$, $n = 1, \ldots, \infty$ such that $\cos x = \sum_{n=1}^{\infty} b_n \sin nx$ for all $x \in (-\pi, \pi)$.

   (b) Every periodic function with period $2L$ is equal to its Fourier series expansion on $[-L, L]$.

   (c) The function $f(x) = x^{1/5}$ is piecewise smooth on the interval $(-1, 1)$. 
(d) Consider a metal rod of length $L$ with cross-sectional area $A$ (figure 1); if $u(x,t)$ represents the temperature in the rod at position $x \in [0,L]$ and time $t > 0$, then the quantity $\int_0^L c(x)\rho(x)u(x,t)A\,dx$ represents the total thermal energy within the rod at time $t$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[thick] (0,0) -- (4,0) arc (180:0:2cm) -- cycle;
\fill[black,opacity=0.5] (0,0) -- (4,0) arc (180:0:2cm) -- cycle;
\draw[thick] (0,0) -- (0,-4); \draw[thick] (4,0) -- (4,-4);
\node at (2,-4) {$A$}; \node at (0,-3) {$x = 0$}; \node at (4,-3) {$x = L$};
\end{tikzpicture}
\caption{Figure 1.}
\end{figure}

(e) Suppose that $f(x)$ is continuous and that $f'(x)$ is piecewise continuous. Then the Fourier series of $f(x)$ can be differentiated term-by-term.
2. (15 points) Consider the function \( f(x) = e^{-3x} \) on \( 0 < x < L \).

(a) (3 pts) Define \( f_{\text{odd}}(x) \), the odd extension of \( f(x) \) on \(-L < x < L\).

(b) (3 pts) Define \( f_{\text{even}}(x) \), the even extension of \( f(x) \) on \(-L < x < L\).

(c) (3 pts) Let \( f_c(x) \) denote the Fourier cosine series of \( f(x) \). What is \( f_c(0) \)?

(d) (3 pts) Let \( f_s(x) \) denote the Fourier sine series of \( f(x) \). What is \( f_s(0) \)?

(e) (3 pts) Find \( b_n \), the coefficients of the Fourier sine series of \( f(x) \).
(workspace)
3. (20 points) Consider the equilibrium temperature distribution of a uniform one-dimensional rod
with sources $Q = K_0x$ of thermal energy, subject to the boundary conditions $u(0) = u(L) = 0$.
(a) (5 pts) Determine the heat energy generated per unit time inside the entire rod.
(b) (10 pts) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and $x = L$.
(c) (5 pts) What relationship should exist between the answers to parts (a) and (b)?
(workspace)
4. (15 points) Determine the equilibrium temperature distribution inside a circular annulus \((R_1 \leq r \leq R_2)\), if the outer radius is at constant temperature \(T_2\) and the inner radius is at constant temperature \(T_1\). The domain is illustrated in figure 2.

![Figure 2.](image-url)
5. (20 points) Use separation of variables or the method of eigenfunction expansion to solve the
Laplace equation $\nabla^2 u = 0$ on the unit square, $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$, subject to the
boundary conditions

\[
\begin{align*}
  u(0, y) &= 1, & u(1, y) &= 2 \sin \pi y \\
  u(x, 0) &= 0, & u(x, 1) &= 0.
\end{align*}
\]
(workspace)