Math 454-001 Winter 2014 Homework 5 due : Thursday, Feb. 27

wave equation

1. page 133, problem 4.2.1, assume $\rho_0(x) = \rho_0 =$ constant.
2. page 142, problem 4.4.3 a
3. page 142, problem 4.4.3 b or problem 4.4.4
   note : This problem is related to the transmission of a signal through a lossy channel. It can be derived from the "telegraph equation," which models telegraph signals through metal wires.
4. page 143, problem 4.4.7 (compare to the d'Alembert solution from class)
5. page 143, problem 4.4.9
6. page 143, problem 4.4.10 a, b (typo in the text, part a should be $u(0,t) = 0$ not $u(0,T) = 0$).

Sturm-Lioville problems

7. For the heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$

where $u = u(x,t)$ with boundary conditions

$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(\pi, t) = 0,$$

do the following :

(a) Use separation of variables to find the associated eigenvalue problem.
(b) Prove that the eigenvalues are real.
(c) Find the eigenvalues and eigenfunctions.
(d) Verify that the eigenfunctions are orthogonal on $[0, \pi]$.
(e) Find $u(x,t)$ if the initial condition is $u(x,0) = f(x)$, where $f(x)$ is given by

$$f(x) = \begin{cases} 0 & 0 \le x \le \frac{9\pi}{20} \\ (\frac{11\pi}{20} - x)(x - \frac{9\pi}{20}) & \frac{9\pi}{20} \le x \le \frac{11\pi}{20} \\ 0 & \frac{11\pi}{20} \le x \le \pi \end{cases}.$$

solution:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n-1/2)^2 t} \sin(n - \tfrac{1}{2})x$$

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^{\pi} f(x)\sin(n-\tfrac{1}{2})x \\ &= \int_{9\pi/20}^{11\pi/20} (\tfrac{11\pi}{20}-x)(x-\tfrac{9\pi}{20})\sin(n-\tfrac{1}{2})x\,dx \end{aligned}$$

$$b_n = \frac{8\sin\frac{\pi k}{4}\left(40\sin\frac{\pi k}{40} - \pi k\cos\frac{\pi k}{40}\right)}{5\pi k^3} \quad \text{where} \quad k = 2n+1$$

8. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0$$

where $u = u(x,t)$ and the following boundary conditions are enforced :

$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(\pi,t) = 0.$$

The string is initially at rest and its initial vertical displacement is $u(x,0) = f(x)$ where $f(x)$ is the same as problem 7.

(a) Find $u(x,t)$.

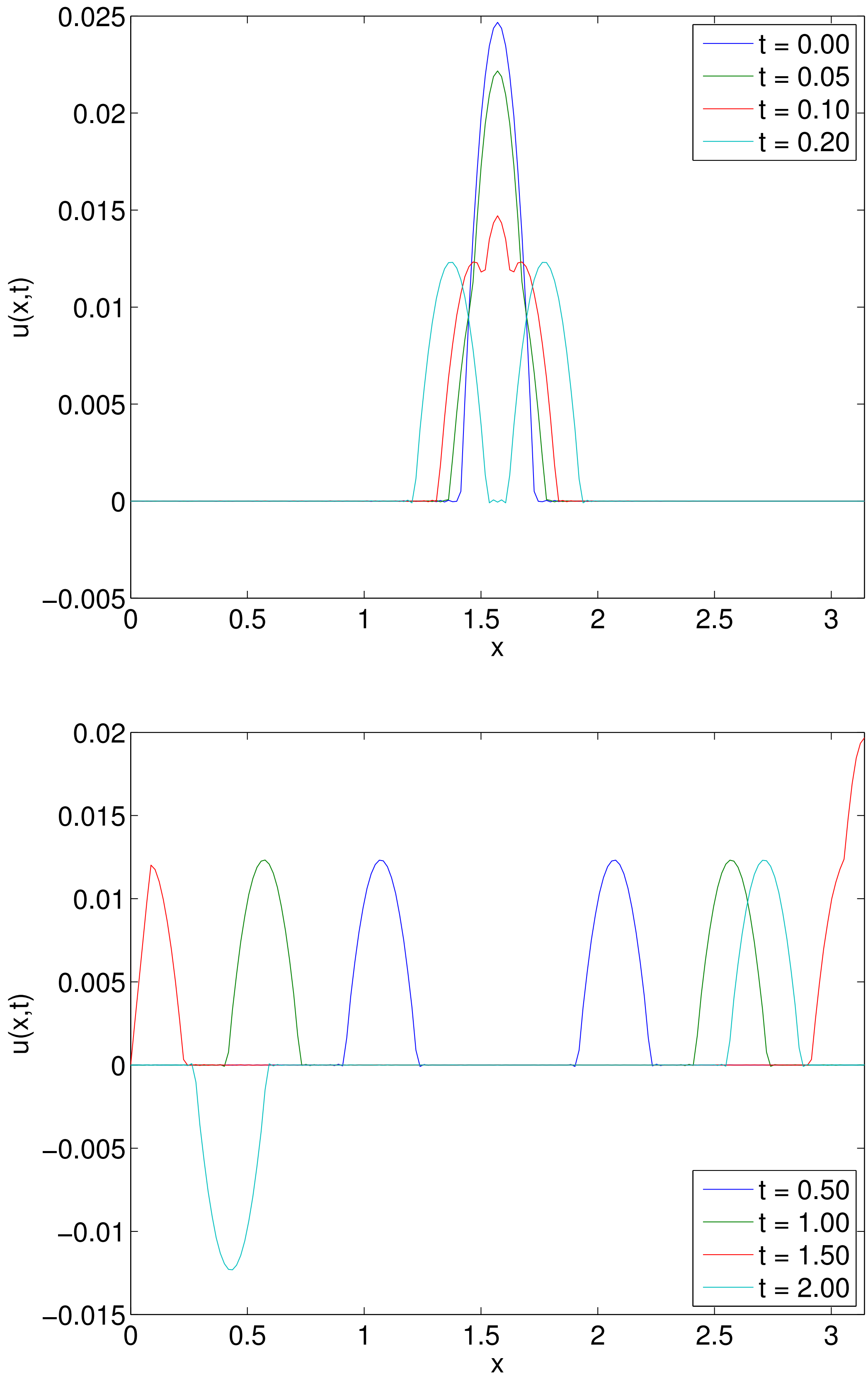
(b) Plot $u(x,t)$ vs. x for $t = 0, 0.05, 0.1, 0.2, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5,$ and 4.0. Make sure to use enough modes to have an accurate solution.

solution

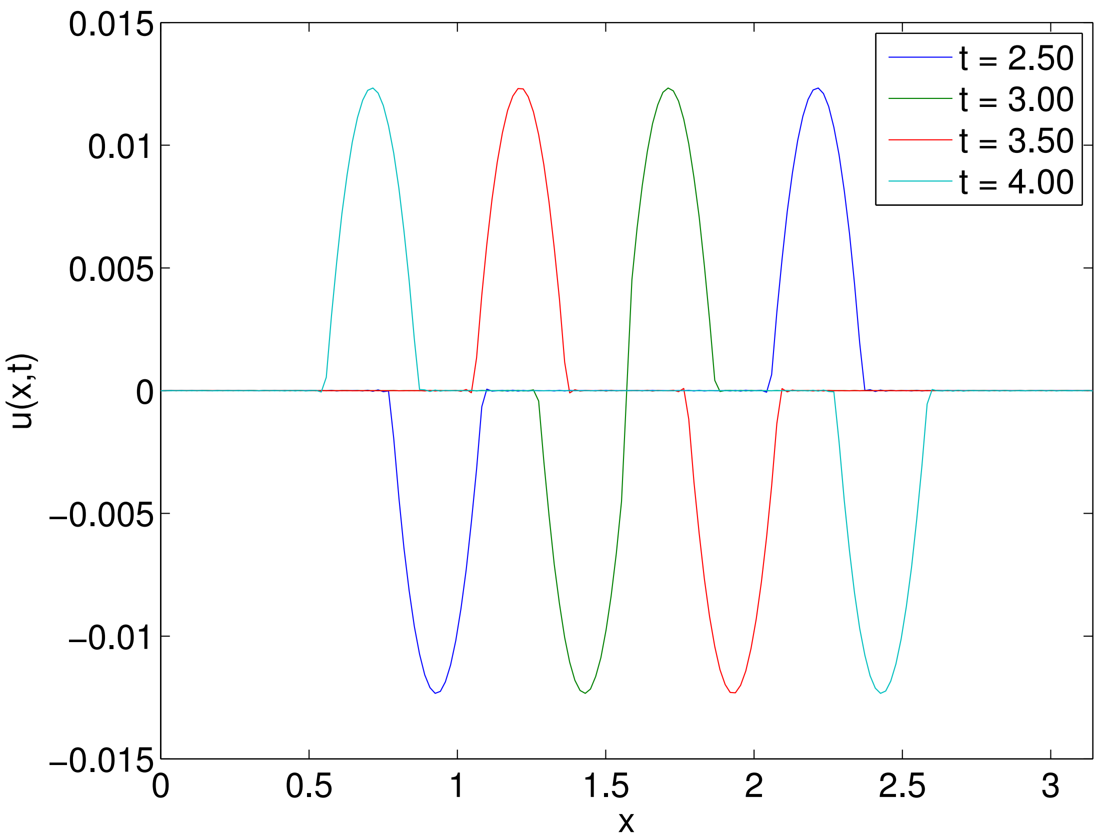
$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos\left((n-\tfrac{1}{2})t\right)\sin\left((n-\tfrac{1}{2})x\right)$$

where b_n are defined in problem 7.

Note that the waves encounter the boundaries at $t \approx 1.5$, and the difference between the fixed boundary condition ($x = 0$) and the free boundary condition ($x = \pi$).



u(x,t)
x
t = 0.00
t = 0.05
t = 0.10
t = 0.20
u(x,t)
x
t = 0.50
t = 1.00
t = 1.50
t = 2.00


0.015
0.01
0.005
0
−0.005
−0.01
−0.015
u(x,t)
0
0.5
1
1.5
2
2.5
3
x
t = 2.50
t = 3.00
t = 3.50
t = 4.00