

Sample Final Exam

1. Answer *each* of the following. 10 points each part:
 - a Given a Hamiltonian $H = H_0 + V$, where H_0 is a Hamiltonian with energy levels $E_n^{(0)}$ (some of which may be degenerate) and V is a perturbation having matrix elements V_{mn} (matrix elements taken with respect to the unperturbed states). What is the validity condition for the energy levels to be given approximately by $E_n = E_n^{(0)} + V_{nn}$? Explain.
 - b Prove that the angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ does not commute with the free particle Dirac Hamiltonian, but that the operator $\mathbf{J} = \mathbf{L} + \hbar\boldsymbol{\sigma}/2$ does commute with it.
 - c A one-dimensional harmonic oscillator has natural frequency ω . The ground state wave function for this oscillator is

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

- At $t = 0$ the oscillator is in its ground state and the frequency ω is changed slowly on a time scale compared with ω^{-1} . If the frequency is specified by $\omega(t)$, write the approximate wave function as a function of time. Explain your reasoning.
- d Estimate the magnetic field strength for which the Zeeman splitting equals the spin-orbit coupling in the $n = 2$ state of hydrogen.
 - e Consider a $J = 1$ ground state to $J = 1$ excited state electric dipole transition. The transition is driven by an electric field polarized parallel to the z axis and there is spontaneous emission from the excited to ground state level with all polarizations. Using properties of the Clebsch-Gordan coefficients, prove that for sufficiently long times all the population ends up in the $m = 0$ ground state level. Hint: you will need the fact that
$$\begin{bmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{bmatrix} = 0 \text{ if } j_1 + j_2 + j_3 \text{ is odd. Explain your reasoning.}$$
 - f What does it mean to say that the angular momentum is the generator of infinitesimal rotations? How can this fact be used to show that, in general, a system possessing spherical symmetry will have degenerate eigenstates?
 - g Consider a hydrogen atom in an external magnetic field. If the Zeeman splitting is much less than the spin orbit coupling, it is often said that ℓ, s, j, m_j are a "good" set of quantum numbers. What is meant by this statement? As the field strength becomes comparable with the spin-orbit coupling, what are the only "good" quantum numbers?
 - h The ground state of oxygen has a $1s^2 2s^2 2p^4$ configuration. In the central field approximation, what is the degeneracy of this state? Give the spectroscopic notation for these states. Explain your reasoning.
 - i What is the WKB approximation? Under what very general conditions is it valid? Explain in *qualitative* terms how it can be used to obtain the transmission coefficient when a particle is incident on a barrier and has energy less than the barrier height. If the energy is greater than the barrier height, what is the transmission coefficient in the WKB approximation? Explain

j Prove that $\frac{(\psi, H\psi)}{(\psi, \psi)} \geq E_0$ for any ψ , where E_0 is the ground state energy of H . How can this result be used to estimate the ground state energy of H ?

2 Consider an electron moving in a 3-dimensional, isotropic harmonic oscillator potential, $V = -e\phi = \mu\omega^2 r^2/2$. In the absence of any spin-orbit coupling and in the absence of an applied magnetic field the eigenstates can be labeled $|n\ell s m_\ell m_s\rangle$ and the eigenenergies are equal to $E_n = (n + 3/2)\hbar\omega$. Recall that $n = 0, 1, 2, \dots$ and $\ell = n, n-2, n-4, \dots$ with $\ell \geq 0$. Now consider the effect of the spin-orbit interaction and the presence of an external magnetic field B aligned along the z axis.

a Using $V_{so} = \frac{1}{2\mu^2 c^2} \frac{1}{r} \frac{dV}{dr} \mathbf{L} \cdot \mathbf{S}$, show that $V_{so} = C\mathbf{L} \cdot \mathbf{S}$, where C is a constant. (5)

b In the absence of a magnetic field, identify the constants of the motion and obtain *exact* expressions for the eigenkets (in terms of the $|n\ell s m_\ell m_s\rangle$) and eigenenergies. (20)

c In a weak magnetic field, *derive* an expression for the energy splitting of the eigenkets. [The expression for the Lande g factor is $g = 1 + \frac{j(j+1)+s(s+1)-\ell(\ell+1)}{2j(j+1)}$. You can use this expression to check the result you obtain, but you are asked to derive the splitting - you cannot use the g factor expression in your derivation]. (40)

d If the limit that the magnetic field interaction is much stronger than the spin-orbit interaction, what are approximate constants of the motion. Neglecting the spin-orbit interaction in this limit, obtain expressions for the eigenkets and eigenenergies. Does the magnetic field lift all the energy degeneracy? Explain. (10)

3

a Three electrons move in a one-dimensional infinite square well of length L . Calculate the ground state wave function and ground state energy, neglecting interactions between the electrons, but including the effects of spin. (20)

b Two electrons move in a three-dimensional, isotropic harmonic potential. In the presence of an applied magnetic field, calculate the energy shift of the ground state of the system. Neglecting interactions between the electrons, but include the effects of spin. (10)

c Two electrons move in a one-dimensional infinite square well of length L . Show that the first excited state of this system is 4-fold degenerate. A perturbation of the form $V(x_1, x_2) = C(x_1 - x_2)^2$ is applied to the system. Calculate the changes in the energy and the eigenstates of the first excited state manifold. Neglecting interactions between the electrons, but include the effects of spin. (25)

4 A particle of mass m and charge q moves in a one dimensional potential $V(x) = m\omega_0^2 x^2/2$. The particle is subjected to an external electric field of the form

$$\mathbf{E} = E\hat{\mathbf{i}}\cos(\omega t).$$

At $t = 0$ the oscillator is in its ground state

a What are $\langle x \rangle$ and $\langle p \rangle$ at $t = 0$? Using these values and the Heisenberg equations of motion,

calculate $\langle x \rangle$ for $t > 0$ without restriction on the value of E . (20)

b Now using time dependent perturbation theory to 1st order in E , calculate $\langle x \rangle$ for $t > 0$. Do **not** make the resonance approximation. (50)

c The correct results of parts (a) and (b) give the same value for $\langle x \rangle$. How can this be the case since (a) is exact and (b) is a perturbation theory result. (10)

[Note: the solution of $\ddot{x} + \omega_0^2 x = F \cos(\omega t)$ is

$$x = A \cos(\omega_0 t + \phi) + F \cos(\omega t) / (\omega_0^2 - \omega^2)$$

$$\langle n|x|n' \rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \left[\sqrt{n'} \delta_{n',n+1} + \sqrt{n} \delta_{n',n-1} \right].$$

5 Consider the Dirac equation for a spin 1/2 particle moving in the potential $q\phi = V(x)$. Assume that an appropriate eigenspinor for the particle can be written as

$$\psi(x, t) = e^{-iEt/\hbar} \begin{pmatrix} F(x) \\ 0 \\ 0 \\ f(x) \end{pmatrix},$$

$E > 0$, and find equations that must be satisfied by $F(x)$ and $f(x)$. In the limit that $E - mc^2 = \delta E \ll mc^2$, show that F satisfies the nonrelativistic, time-independent Schrödinger equation with energy δE . (25)

Equations

$$U_R(\boldsymbol{\theta}) = e^{-i\mathbf{J}\cdot\boldsymbol{\theta}/\hbar}$$

$$U_R(\alpha, \beta, \gamma) = e^{-iJ_z\alpha/\hbar} e^{-iJ_y\beta/\hbar} e^{-iJ_z\gamma/\hbar}$$

$$E_n \sim E_n^{(0)} + \langle n|V|n\rangle + \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$|n\rangle \sim |n\rangle^{(0)} + \sum_{m\neq n} \frac{\langle m|V|n\rangle}{E_n^{(0)} - E_m^{(0)}} |m\rangle^{(0)}$$

$$e^{-i\mathbf{n}\cdot\boldsymbol{\sigma}\theta} = \cos(\theta)\mathbf{1} - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin(\theta)$$

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$V = -\mathbf{m}\cdot\mathbf{B}; V = -\mathbf{p}\cdot\mathbf{E}$$

$$\mathbb{R}(\boldsymbol{\epsilon}) = \begin{pmatrix} 1 & -\epsilon_z & \epsilon_y \\ \epsilon_z & 1 & -\epsilon_x \\ -\epsilon_y & \epsilon_x & 1 \end{pmatrix}$$

$$\mathbf{D}_{mm'}^{(j)}(\alpha, \beta, \gamma) = \langle jm|U_R(\alpha, \beta, \gamma)|jm'\rangle; \quad (T_k^q)' = \sum_{q'} \mathbf{D}_{q'q}^{(j)}(\alpha, \beta, \gamma) T_k^{q'};$$

$$u = \tilde{\rho}_{\uparrow\uparrow} + \tilde{\rho}_{\downarrow\downarrow}; v = i(\tilde{\rho}_{\uparrow\downarrow} - \tilde{\rho}_{\downarrow\uparrow}); w = \rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow}; \rho_{\uparrow\downarrow} = \tilde{\rho}_{\uparrow\downarrow} e^{-i\omega t}$$

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_x \\ 0 \\ \delta \end{pmatrix}$$

$$\rho_{ab} = ab^*$$

$$d\mathbf{U}/dt = \boldsymbol{\Omega} \times \mathbf{U}; \quad \omega_x = \frac{-gqB_x}{4\mu c}; \quad \delta = \omega_0 - \omega$$

$$\mathbf{m} = \frac{q\mathbf{L}}{2\mu c}; \quad \mathbf{m} = \frac{gq\mathbf{S}}{2\mu c}; \quad \beta_0 = \frac{e\hbar}{2\mu c} \quad g = 2 \text{ for electron}$$

$$\langle \alpha jm|T_k^q|\alpha' j' m'\rangle = \frac{1}{\sqrt{2j+1}} \begin{bmatrix} j' & k & j \\ m' & q & m \end{bmatrix} \langle \alpha j||T_k^q||\alpha j\rangle$$

$$|j_1 j_2 jm\rangle = \sum_{m_1 m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |j_1 m_1\rangle |j_2 m_2\rangle$$

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{jm} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |j_1 j_2 jm\rangle$$

$$[A, BC] = B[A, C] + [A, B]C$$

$$[J_x, y] = i\hbar z; \quad [J_x, T_k^q] = \hbar q T_k^q; \quad [J_{\pm}, T_k^q] = \hbar \sqrt{(k \mp q)(k \pm q + 1)} T_k^{q\pm 1}; \quad J_{\pm} = J_x \pm iJ_y$$

$$\psi(\mathbf{r})' = U_R \psi(\mathbf{r}) = \psi(R^{-1}\mathbf{r}); \quad |jm\rangle' = U_R |jm\rangle = \sum_{m'} \mathbf{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) |jm'\rangle$$

$$\begin{bmatrix} j_1 & \frac{1}{2} & j \pm \frac{1}{2} \\ m - \frac{1}{2} & \frac{1}{2} & m \end{bmatrix} = \pm \sqrt{\frac{j_1 \pm m + \frac{1}{2}}{2j_1 + 1}}; \quad \begin{bmatrix} j_1 & \frac{1}{2} & j \pm \frac{1}{2} \\ m + \frac{1}{2} & -\frac{1}{2} & m \end{bmatrix} = \sqrt{\frac{j_1 \mp m + \frac{1}{2}}{2j_1 + 1}}$$

$$J_{\pm} |jm\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle$$

$$a = (\xi + i\eta)/\sqrt{2}; \quad a|n\rangle = \sqrt{n}|n-1\rangle; \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\rho = \frac{1}{N} \sum_{j=1, N} \rho^{(j)}$$

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{bmatrix} = 0 \text{ if } j_1 + j_2 + j_3 \text{ is odd}$$

$$[J_\pm, T_k^q] = \hbar \sqrt{(k \mp q)(k \pm q + 1)} T_k^{q\pm 1}; \quad [J_z, T_k^q] = \hbar q T_k^q$$

$$(T_k^q)' = \sum_{q'} \mathbf{D}_{q'q}^{(k)}(\alpha, \beta, \gamma) T_k^{q'}$$

$$J_\pm = J_x \pm iJ_y$$

$$T_1^{\pm 1} = \mp \frac{x \pm iy}{\sqrt{2}}; \quad T_1^0 = z$$

$$V_{so} = \frac{1}{2m^2 c^2} \frac{1}{r} \frac{dV}{dr} \mathbf{L} \cdot \mathbf{S}$$

$$\frac{(\psi, H\psi)}{(\psi, \psi)} \geq E_0$$

$$P_{E_f}(t) = \frac{2\pi}{\hbar} |V_{fi}(E_i)|^2 \rho(E_i) t$$

$$P_{E_f}(t) = \frac{2\pi}{\hbar} |V_{fi}(E_i + \hbar\omega)|^2 \rho(E_i + \hbar\omega) t$$

$$H = H_{cf} + V_1 + V_2; \quad H_{cf} = \sum_i \left(\frac{p_i^2}{2m} + V_c(r_i) \right);$$

$$V_1 = - \sum_i \left(V_c(r_i) + \frac{Ze^2}{r_i} \right) + \sum_{i < j} \frac{e^2}{|r_i - r_j|}; \quad V_2 = \sum_i \frac{1}{2mc^2} \frac{1}{r_i} \frac{dV_c}{dr_i} \mathbf{l}_i \cdot \mathbf{s}_i$$

$$i\hbar \frac{\partial \psi}{\partial t} = (\mathbf{c}\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2) \psi$$

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma_0 = \gamma^0$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2g_{ij}$$

$$\alpha_i \beta + \beta \alpha_i = 0$$

$$i\hbar \frac{\partial \psi}{\partial t} = \left(\mathbf{c}\boldsymbol{\alpha} \cdot \left(\mathbf{p} - \frac{q\mathbf{A}}{c} \right) + \beta mc^2 + q\phi \right) \psi$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^i = \gamma^0 \alpha_i$$

$$(i\hbar \gamma^\mu \partial_\mu - mc) \psi = 0$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\left(i\hbar \gamma^\mu \left(\partial_\mu - \frac{qA_\mu}{c} \right) - mc \right) \psi = 0$$

$$C_n^p = \binom{n}{p} = \frac{n!}{p!(n-p)!}$$