

Problem Set 9

Due Friday, March 19

- 1 Derive the transmission coefficient for barrier penetration in the WKB approximation..
2. Obtain the WKB bound state energies for the 3-D isotropic harmonic oscillator and the hydrogen atom. You can use the WKB condition for bound state energies as

$$\int_b^a \left[\frac{2\mu}{\hbar^2} \left(E - V(r) - \frac{\hbar^2(\ell + \frac{1}{2})^2}{2\mu r^2} \right) \right]^{1/2} dr = \left(n + \frac{1}{2} \right) \pi.$$

Compare the results with the exact values.

3. In class it was shown that the WKB approximation for the phase shifts is given by

$$\delta_\ell^{WKB} = \frac{1}{2} \left(\ell + \frac{1}{2} \right) \pi + \int_{r_c}^{\infty} (k_\ell - k_0) dr - k_0 r_c$$

where

$$k_\ell = \left[\frac{2\mu}{\hbar^2} \left(E - V(r) - \frac{\hbar^2(\ell + \frac{1}{2})^2}{2\mu r^2} \right) \right]^{1/2}; \quad k_0 = \left[\frac{2\mu}{\hbar^2} E \right]^{1/2}$$

and r_c is the point of closest approach for a given value of ℓ and E . Prove that

$$\delta_\ell^{WKB} = \frac{\pi}{2} \left(\ell + \frac{1}{2} \right) - \int_{r_c}^{\infty} r \frac{dk_\ell}{dr} dr.$$

In the partial wave expansion for the scattering amplitude, expand the Legendre polynomial as

$$P_\ell(\cos\theta) \sim \left[\frac{2}{\pi(\ell + \frac{1}{2}) \sin\theta} \right]^{1/2} \cos \left[\left(\ell + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right]$$

(valid for large ℓ) to prove that, if one considers the major contribution to the sum over partial waves comes from the point of stationary phase, the major contribution to the sum comes from a deflection angle

$$\Theta = \pi - 2(J/\hbar) \int_{r_c}^{\infty} \frac{dr}{r^2 \left[\frac{2\mu}{\hbar^2} \left(E - V(r) - \frac{J^2}{2\mu r^2} \right) \right]^{1/2}}$$

where $J = (\ell + \frac{1}{2})\hbar$. This is just the classical result.

4. A Hamiltonian is of the form $H = H_0 + V(t)$, where

$$H_0 = \frac{\hbar}{2} \begin{pmatrix} -\omega & 0 \\ 0 & -\omega \end{pmatrix}; \quad V(t) = \frac{\hbar}{2} \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & -\alpha(t) \end{pmatrix}$$

Write the differential equations for the state amplitudes in the "normal" and interaction

representations.

5-6. Consider a spin 1/2 atom in a magnetic field of the form

$$\mathbf{B}(t) = B_0[\cos(\theta)\mathbf{k} + \sin(\theta)\mathbf{i}],$$

where

$$\theta(t) = \frac{\pi}{2}(1 - e^{-t/T}).$$

The Hamiltonian is

$$H = \beta_0 B_0[\cos(\theta)\sigma_z + \sin(\theta)\sigma_x].$$

At $t = 0$, the system is the state spin "down".

Write the differential equations for the spin up and down state amplitudes. Take $T = 1$, such that the frequency $\omega = \beta_0 B_0/\hbar$ is measured in units of $1/T$. Solve these equations numerically and plot $|a_{\downarrow}(t)|^2$ as a function of t for $\omega = 3, 0 \leq t \leq 10$; $\omega = 10, 0 \leq t \leq 7$; $\omega = 0.1, 0 \leq t \leq 100$.

In the case of $\omega = 0.1$, compare your answer with the prediction of the sudden approximation. Why should it be good?

In the case of $\omega = 10$, compare your answer with the prediction of the adiabatic approximation (plot both solutions on the same graph). Why should it be good? What is the origin of the wiggles in the exact solution? Why do they appear?

7. **OPTIONAL** To eliminate the wiggles in Prob. 5-6, take

$$\theta(t) = \frac{1}{2} \left(\frac{\pi}{2} + \arctan(t/T) \right)$$

for $T = 1$ and $\omega = 10$ with initial conditions $a_{\downarrow}(-100) = 1$; $a_{\uparrow}(-100) = 0$. Solve the equations numerically and plot $|a_{\downarrow}(t)|^2$ as a function of t for $-10 \leq t \leq 10$. Compare with the adiabatic solution.