Phys 512 Prof. P. Berman
Quantum Mechanics II Winter,2004

Problem Set 3
Due Friday, January 30

1. Consider a spin 1/2 atom in a state $|\chi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ in a system quantized along the z-axis. Suppose there is a magnetic field **B** having polar angles (θ,ϕ). Rotate the system so the magnetic field is along the z-axis and calculate the components of $|\chi\rangle'$ after this active rotation [Hint: show that the active rotation is given by Euler angles $(\alpha,\beta,\gamma) = (\pi/2,-\theta,-\phi)$]. If $a = 1$ and $b = 0$ show that the result agrees with that given on page 180 of the coursepack notes with $\phi = \pi/2, \theta \to \phi$.

2-3. Consider a spin one particle having standard components

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that for an infinitesmal rotation around the z-axis, that the components of **S** transform as a vector (this is true for any 3×3 matrices obeying the standard commutation relations). On the other hand, show that the components of the spin state

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix}$$

do not transform as a cartesian vector. Thus, in the standard $|sm\rangle$ basis, the components of ψ do not transform as a cartesion vector. However, there is a basis in which the components of ψ *do* transform as a cartesian vector. To see this explicitly show that under the unitary transformation

$$\psi' = \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix}$$

the spin **S** transforms as

$$S'_x = i\hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad S'_y = i\hbar\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad S'_z = i\hbar\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and that the components of ψ' transform as a vector under the infinitesmal rotation $\mathbf{U} = \exp[-i\mathbf{S}' \cdot \boldsymbol{\delta\varphi}/\hbar]$. Actually, both ψ' and ψ transform as vectors under rotation. It is just that different definitions of the transformation properties are used for standard and cartesian components of a vector.

4. Consider the 2-D isotropic harmonic oscillator with Hamiltonian $H = \frac{\hbar\omega}{2}(\eta^2 + \xi^2)$, where η and ξ are dimensionless variable satisfying $[\xi_x, \eta_x] = i$; $[\xi_y, \eta_y] = i$. Prove that the Hamiltonian can be written in the form

$$H = \hbar\omega\sqrt{(4K^2 + 1)}\,,$$

where

$$\begin{aligned} K_1 &= \frac{\eta_x^2 - \eta_y^2}{4} + \frac{\xi_x^2 - \xi_y^2}{4}; \\ K_2 &= \frac{\eta_x\eta_y}{2} + \frac{\xi_x\xi_y}{2}; \\ K_3 &= \frac{\xi_x\eta_y - \xi_y\eta_x}{2}. \end{aligned}$$

Show that the $K's$ satisfy the standard commutation relations for angular momentum matrices. Use this fact to obtain the allowed energy levels for the oscillator and to prove that the underlying symmetry group is $SU(2)$. How does this explain the degeneracy of the levels?

5. Use the program on page 299 of the coursepack to calculate the rotation matrices for J=1/2, J=1, and J=3/2.