Phys 511 — Prof. P. Berman
Quantum Mechanics I — Fall, 2003

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Sample Midterm Exam Time - 90 minutes.

1. Explain *each* of the following (one or two sentences will do) 10 points each part:

a Although the parity operator commutes with the Hamiltonian for a free particle, $\exp(i\mathbf{p}\cdot\mathbf{r}/\hbar)$ is *not* a simultaneous eigenfunction of the Hamiltonian and the parity operator.

b A particle moves in a symmetric infinite square well. Although $\langle x\rangle = 0$ for each eigenstate in the well, $\langle x\rangle \neq 0$ for a particle in a superposition of eigenstates.

c It takes many measurements on identically prepared systems to construct $|\psi(\mathbf{r},t)|^2$.

d Matter can exhibit both wave-like and particle-like behavior.

2 The simple harmonic oscillator (in dimensionless variables) has normalized eigenfunctions

$$\psi_n(\xi) = \frac{1}{\sqrt{\pi^{1/2}2^n n!}}e^{-\xi^2/2}H_n(\xi)$$

where the first few Hermite polynomials are

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2\xi \\ H_2(x) &= -2+4\xi^2 \\ H_3(x) &= -12\xi+8\xi^3 \end{aligned}$$

and the dimensionless energies are $E_n = \left(n+\frac{1}{2}\right)$.

Consider a particle moving in a potential $V(x) = \xi^2/2$ (in dimensionless variables) for $x > 0$; $V(x) = \infty$ for $x \leq 0$. At $t = 0$ the particle is in the state described by

$$\begin{aligned} \psi(\xi,0) &= A\frac{1}{\sqrt{\pi^{1/2}}}e^{-\xi^2/2}\left[\left(\sqrt{2}-2\sqrt{3}\right)\xi + \frac{4}{\sqrt{3}}\xi^3\right] \text{ for } \xi > 0 \\ &= 0 \text{ for } \xi \leq 0 \end{aligned}$$

where A is chosen such that $\int_{-\infty}^{\infty}|\psi(\xi,0)|^2 d\xi = 1$.

a Find the eigenfunctions and eigenvalues for the particle moving in this potential. 10 points

b Show that $\psi(\xi, 0) = \frac{\sqrt{2}}{\sqrt{5}}[\psi_1(\xi) + 2\psi_3(\xi)]$ for $\xi > 0$, where $\psi_1(\xi)$ and $\psi_3(\xi)$ are given above. 10 points

c Calculate the probability that a measurement of the oscillator's energy yields a value $E = 3/2$ at $t = 0$? What is $\langle E \rangle$? 10 points

d Write an integral expression for $\left\langle \hat{\xi} \right\rangle$ at $t = 0$ (do **not** evaluate the integrals). Would $\left\langle \hat{\xi} \right\rangle = 0$ in an eigenstate? Explain. 10 points

e Prove that $\langle \hat{\eta} \rangle = 0$ at $t = 0$ where $\hat{\eta}$ is the dimensionless momentum operator. Hint: You do not need to evaluate any integrals to prove this if you use the definition of $\hat{\eta}$ and an integration by parts. 10 points

f Write an expression for the wave function for times $t > 0$. (you may use units in which $\hbar = 1$). 10 points

Note: $\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n a^n} \sqrt{\frac{\pi}{a}}$ for $n \geq 1$; $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$; $\int_0^{\infty} x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}$ for $n \geq 0$.
$\frac{d\psi_n}{d\xi} = \sqrt{2n}\psi_{n-1} - \xi\psi_n$; $\quad \xi\psi_n = \left(\sqrt{n+1}\psi_{n+1} + \sqrt{n}\psi_{n-1}\right)/\sqrt{2}$

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a Prove that $\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$. 10 Points

b For any time independent Hermitian operator $\hat{A}$, prove that $i\hbar \frac{d\langle \hat{A} \rangle}{dt} = \left[\hat{A}, \hat{H}\right]$, where $\hat{H}$ is the Hamiltonian describing the system in which the operator $\hat{A}$ is defined. 10 points

c Now consider a free particle of mass m in one dimension having $\langle x \rangle = 0$ at $t = 0$. Find $\langle x^2 \rangle$ as a function of t in terms of $a \equiv \langle p^2 \rangle_{t=0}$, $b \equiv \langle xp + px \rangle_{t=0}$, and $c \equiv \langle x^2 \rangle_{t=0}$. Find the time for which $\langle x^2 \rangle$ is a minimum and show that the wave function must spread for sufficiently long times. Hint: Use the results of parts (a) and (b). 30 points

4 A particle of mass m is confined to the potential $V = \infty$, $|x| > b$; $V = 0$, $|x| \leq (b - a)$; $V = -V_0$, $|x| \leq a$ (see figure).

a Using a simple uncertainty principle argument, prove that there is not necessarily a bound state for energies $E < 0$. Why does this result differ from that found in class for a one-dimensional finite well for which a bound state always exists. 15 points

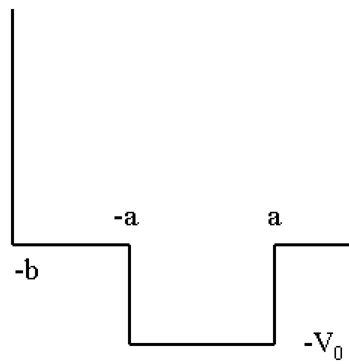

Figure 1:

b Now write down the eigenfunctions and obtain an equation that can be used to obtain the eigenvalues for the lowest energy eigenstate when $E < 0$. (Hints: What must be the wave function for $|x| > b$? What is the parity of the lowest energy eigenstate? The boundary condition at $|x| = b$ can be built directly into the wave function, although this is not necessary.) 25 points

c **This is a 25 point bonus question** For the case $b = 2a$, assume that $\beta < \pi/2$ $\left(\beta^2 = \frac{2mV_0a^2}{\hbar^2}\right)$ and show graphically that the condition for which a bound state exists is determined by a solution of the equation $\tan\beta \geq 1/\beta$.