Phys 511 Prof. P. Berman
Quantum Mechanics I Fall, 2003

Sample Final Examination

Answer all questions. **Read all questions carefully and answer what is asked for.** Show all work and write legibly. Credit as indicated. Time: 3 hours.

I. Answer *each* of the following (15 points each part, except for part 11 which carries 25 points credit):

1. State at least *three* properties of Hermitian operators.
2. Of what use are commuting operators in quantum theory? Give an example.
3. Calculate the reflection coefficient for scattering from a potential step barrier when the energy is greater than the height of the barrier V_0.
4. A particle of mass m moves in a 1-D infinite square well potential, $V(x) = 0; 0 \leq x \leq L$; $V(x) = \infty$; otherwise. If $\psi(x,0) = \frac{1}{\sqrt{b-a}}$; $a \leq x \leq b$ and is zero otherwise $(0 < a, b < L)$, find $\psi(x,t)$.
5. For potentials with finite range, prove that the asymptotic form for the radial wave function for positive energy solutions as $r \sim \infty$ is of the form $u_\ell = A_\ell \sin(kr + \beta_\ell)$. How is β_ℓ related to the partial wave shifts? Explain.
6. Estimate the ground state energy of a particle of mass m moving in the potential $V(x) = mg|x|$.
7. Write the operator L_x in the $|\ell m\rangle$ basis for a state with $\ell = 2$. What are the eigenvalues of this operator in this $\ell = 2$ subspace? In order to get the eigenkets, what physical operation must be carried out on the $|\ell m\rangle$ eigenkets?
8. An operator A whose eigenkets are denoted by $|b\rangle$ can be expressed in the "$|b\rangle$" basis as
$$A = 4|b_1\rangle\langle b_1| + 4|b_2\rangle\langle b_2| + |b_1\rangle\langle b_2| + |b_2\rangle\langle b_1|.$$
Find the eigenvalues of this operator and express its eigenvectors in the $|b\rangle$ basis.
9. A particle of mass m moves in a 1-D infinite square well potential, $V(x) = 0; 0 \leq x \leq L$; $V(x) = \infty$; otherwise. At $t = 0$, the wave function for the particle is smooth, is centered in the well, and represents a superposition of the first 10 energy states of the well. Estimate Δx at $t = 0$. Estimate the time it will take for the wave function to spread to the walls of the potential.
10. At $t = 0$, a 1-D simple harmonic oscillator is in the state $\psi(x,0) = A[\psi_0(x) + 2\psi_2(x) + \psi_1(x)]$, where $\psi_n(x)$ is a normalized eigenfunction and A is a normalization factor. The natural frequency of the oscillator is ω. What are possible values that a single measurement of the energy can yield and what is the probability for each measurement? What is the variance $\langle E^2\rangle - \langle E\rangle^2$ and what is the physical significance of this quantity?
11. At $t = 0$, the normalized state of a 1-D simple harmonic oscillator is $\psi(\xi,0) = \frac{1}{\sqrt{6}}[\psi_0(\xi) + 2\psi_2(\xi) + \psi_1(\xi)]$, where ξ is a dimensionless variable. Calculate $\langle \xi^2 \rangle$ at time t if the natural frequency of the oscillator is ω. (25)

II. A particle of mass μ moves in the potential $V(r) = -V_0 e^{-r^2/a^2}$, where $V_0 > 0$.

1. Sketch the potential as a function of r. (5)
2. Draw the effective potential for $\ell = 0$ and $\ell > 0$. Calculate the condition for which is there a relative maximum in the effective potential? (20)
3. For $E < 0$, what determines the number of possible bound states? Is there always at least one bound state? Assuming there are several bound states **and without solving the radial equation**, sketch the general form for the ground state and the two lowest energy excited state wave functions. **Explain your reasoning in obtaining your graphs**. (20)
4. For $E > 0$, determine the conditions for which bound states can occur **classically**. In the quantum case, characterize the scattering as a function of energy. Under what conditions can there be resonances in the scattering cross section? Can there be bound states? (20)

III. Consider scattering of a particle of mass μ by the potential

$$V = V_0 \quad r \leq a;$$
$$V = 0 \quad r > a,$$

where V_0 is a positive constant.

1. In the limit of low energy scattering $ka \ll 1$, use the partial wave expansion to calculate the differential scattering cross section $\left(k^2 = \frac{2\mu}{\hbar^2}E\right)$. *How does your result differ fundamentally than in the case of an attractive potential*? (30)
2. For arbitrary ka, obtain the equations from which the partial wave shifts can be obtained. About how many partial waves are needed to get a good value for the scattering amplitude? Explain. (20)
3. Calculate the scattering amplitude and differential scattering cross section in first Born approximation. Under what conditions will the result agree with part (1)? Explain (20)

IV. The Laplacian in cylindrical coordinates in two dimensions is

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2},$$

where $\rho = \sqrt{x^2 + y^2}$; $\cos\varphi = x/\rho$. Consider the Schrödinger equation in two dimensions for a cylindrically symmetric potential $V(\rho) = -A/\rho$ where $A > 0$. A particle of mass μ moves in this potential. Consider only negative energy solutions with $k^2 = -\frac{2\mu}{\hbar^2}E > 0$

1. Show that an eigenfunction can be written as

   $$\psi_{Em}(\rho,\varphi) = R_{Em}(\rho)e^{im\varphi},$$

   where m is an integer (positive, negative or zero), and $R_{Em}(\rho)$ satisfies the radial equation

   $$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR_{Em}}{d\rho}\right) + \frac{2\mu}{\hbar^2}\left(E + \frac{A}{\rho} - \frac{m^2}{\rho^2}\right)R_{Em} = 0. \qquad (10)$$

2. Show that the asymptotic form of the wave function is $\rho^{|m|}$ for $\rho \sim 0$ and $e^{-|k|\rho}$ for $\rho \sim \infty$? (15)
3. Find the **bound state** energies, (unnormalized) bound state wave functions, and the degeneracy of each bound state. (50)

   The following relations may save you some time:
   If $g(x) = x^m e^{-kx} f(x)$, then
   $g'(x) = [(\frac{m}{x} - k)f + f']x^m e^{-kx}$ and
   $g''(x) = \left[\left(\frac{m^2}{x^2} - \frac{m}{x^2} - \frac{2km}{x} + k^2\right)f + 2(\frac{m}{x} - k)f' + f''\right]x^m e^{-kx}$

Some Useful Expressions

Laguerre's equation is $xf''(x) + (\alpha + 1 - x)f' + qf = 0,$ which, for integral q has solutions which are the Laguerre polynomials $L_q^\alpha(x)$.

$m_e c^2 = 0.511$ Mev
$\lambda_c = h/mc$
$\alpha = e^2/\hbar c$
$h = 6.63 \times 10^{-27}$erg•s$= 4.14 \times 10^{-15}$eV•s
$\hbar = 1.06 \times 10^{-27}$erg•s$= 0.66 \times 10^{-15}$eV•s
$f_k(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell + 1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)$
$\sigma_k(\theta) = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell + 1)\sin^2\delta_\ell$
$j_\ell(kr) \sim \sin(kr - \frac{\ell\pi}{2})/kr$
$n_\ell(kr) \sim -\cos(kr - \frac{\ell\pi}{2})/kr$
$h_\ell^{(1)}(kr) \sim -i\exp[i(kr - \frac{\ell\pi}{2})]/kr$
$h_\ell^{(2)}(kr) \sim i\exp[-i(kr - \frac{\ell\pi}{2})]/kr$
$j_0(kr) = \sin(kr)/kr$
$j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}$
$n_0(kr) = -\cos(kr)/kr$
$n_1(kr) = -\frac{\cos(kr)}{(kr)^2} - \frac{\sin(kr)}{kr}$
$h_0^{(1)}(kr) = -i\exp(ikr)/kr$
$h_0^{(2)}(kr) = i\exp(-ikr)/kr$
$f_q(\theta) = -\frac{2\mu}{\hbar^2 q}\int_0^\infty rV(r)\sin(qr)dr;\ \ q = 2k\sin(\theta/2)$
$u_\ell'' + \left(k^2 - \tilde{V}(r) - \frac{\ell(\ell+1)}{r^2}\right)u_\ell = 0$

$\int_0^a y\sin(qy)dy = \frac{1}{q^2}[\sin(qa) - qa\cos(qa)];$

$L_\pm|\ell,m\rangle = \sqrt{(\ell \mp m)(\ell \pm m + 1)}\,\hbar|\ell,m \pm 1\rangle$
$L_\pm = L_x \pm L_y$
$a = (\xi + i\eta)/\sqrt{2}\,;\quad a^\dagger = (\xi + i\eta)/\sqrt{2}$
$a|n\rangle = \sqrt{n}\,|n-1\rangle;\qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b};\qquad \tan(\epsilon) \sim \epsilon + \frac{\epsilon^3}{3};\quad \tanh(\epsilon) \sim \epsilon - \frac{\epsilon^3}{3}$