Introductory Quantum Mechanics: A Traditional Approach Emphasizing Connections with Classical Physics

Supplementary Material

Supplementary material will be added from time to time.

Sec. 5.3 - Collapse of the Wave Function

My view that the collapse of the wave function is not a very meaningful concept is consistent with the *statistical interpretation* or *ensemble interpretation* of quantum mechanics expressed in the articles of L. B. Ballentine [*The statistical interpretaion of quantum mechanics,* Reviews of Modern Physics **42**, 358-381 (1970)] and R. G. Newton [*Probability interpretation of quantum mechanics,* American Journal of Physics **48**, 1029-1034 (1980)]. In some sense, these articles express the sentiment that quantum mechanics provides a probability interpretation of measurement only when carried out on an enesemble of identically prepared systems.

Single-Photon States

In discussing experiments on Bell's theorem, I introduced the concept of a single photon state, such as that emitted in spontaneous emission from a single excited atom. This quantum state is *fundamentally different* from that associated with the output of a laser that has been sent through a series of neutral density filters to reduce its intensity to some arbitrarily small value. That field is still a *coherent state* of the field whose average energy (for a pulse) can be much less than the energy $\hbar\omega_0$, where ω_0 is the carrier frequency of the pulse. Authors often say that the field is so weak that it serves as a single photon source, but this is not true. A single photon source can entangle two atoms, but a low intensity coherent state field cannot. The two types of fields have very different second-order correaltion functions.

As an interesting example, consider a weak field pulse and a single-photon field incident on a thin film of glass. In both cases, there is a transmitted and reflected pulse, but only in the single-photon state are you guaranteed that the transmitted and reflected pulses are perfectly anti-correlated in the sense that it is impossible in a single experiment to measure photo-signals on detectors placed on both sides of the glass film.

Derivation of Equation (21.129)

To derive Eq. (21.129), I used a stated that Eq. (21.62) could be regarded as a vector equation of the form

$$\hat{\mathbf{A}} = \frac{\langle \alpha', J, m_J | \, \hat{\mathbf{J}} \cdot \hat{\mathbf{A}} \, |\alpha, J, m_J\rangle}{\hbar^2 J\,(J+1)} \hat{\mathbf{J}},$$

provided diagonal matrix elements of both sides are taken. I then took the scalar product of this equation with $\hat{\mathbf{I}}$ and $\hat{\mathbf{A}} = \hat{\mathbf{L}}$ or $\hat{\mathbf{S}}$ and evaluated matrix elements of the scalar products. This produces the correct result.

A more rigorous proof is a direct one.

$$A_{n\ell jf} = \langle n,\ell,j,f,0| \frac{\left(\hat{\mathbf{L}}-\hat{\mathbf{S}}\right)\cdot\hat{\mathbf{I}}}{r^3} |n,\ell,j,f,0\rangle \tag{1}$$

In Eq. (1), I replace $(\mathbf{L}-\mathbf{S})\cdot\mathbf{I}$ by

$$(\mathbf{L}-\mathbf{S})\cdot\mathbf{I} = \sum_{q=-1}^{1}(-1)^q\left(L_1^q - S_1^q\right)I_1^{-q}, \tag{2}$$

where

$$G_1^1 = -\frac{G_+}{\sqrt{2}} = -\frac{G_x+iG_y}{\sqrt{2}}; \tag{3a}$$

$$G_1^{-1} = \frac{G_-}{\sqrt{2}} = \frac{G_x-iG_y}{\sqrt{2}}; \tag{3b}$$

$$G_1^0 = G_z \tag{3c}$$

$(G = L, S, I)$ are components of an irreducible tensor operator of rank 1. Using the properties of any angular momentum operator G

$$G_\pm|g,m_g\rangle = \hbar\sqrt{(g\mp m_g)(g\pm m_g+1)}\,|g,m_g\pm1\rangle; \tag{4a}$$

$$G_z|g,m_g\rangle = m_g\hbar\,|g,m_g\rangle, \tag{4b}$$

I can evaluate Eq. (1) as

$$\begin{aligned} A_{n\ell jf} &= \langle r^{-3}\rangle \sum_{m_I,m_s=-1/2}^{1/2}\sum_{q,=-1}^{1}(-1)^q \begin{bmatrix} \ell & 1/2 & j \\ -m_I-m_s & m_s & -m_I \end{bmatrix}\begin{bmatrix} j & 1/2 & f \\ -m_I & m_I & 0 \end{bmatrix} \\ &\times \begin{bmatrix} \ell & 1/2 & j \\ -m_I'-m_s' & m_s' & -m_I' \end{bmatrix}\begin{bmatrix} j & 1/2 & f \\ -m_I' & m_I' & 0 \end{bmatrix} \\ &\times \left(L_{q,-m_I'-m_s',-m_I-m_s} - S_{qm_s'm_s}\right) I_{qm_I'm_I}, \end{aligned} \tag{5}$$

where

$$L_{qm_\ell'm_\ell} = \langle Lm_\ell'|\hat{L}_1^q|Lm_\ell\rangle = \hbar\begin{bmatrix} m_\ell\delta_{q,0}\delta_{m_s,m_s'} \\ +\frac{1}{\sqrt{2}}\begin{pmatrix} -\delta_{q,1}\sqrt{(g-m_g)(g+m_g+1)}\delta_{m_\ell',m_\ell+1} \\ +\delta_{q,-1}\sqrt{(g+m_g)(g-m_g+1)}\delta_{m_\ell',m_\ell-1} \end{pmatrix}\end{bmatrix} \tag{6a}$$

$$S_{qm_s'm_s} = \langle Sm_s'|\hat{S}_1^q|Sm_s\rangle = \hbar\begin{bmatrix} m_s\delta_{q,0}\delta_{m_s,m_s'} \\ +\frac{1}{\sqrt{2}}\left(-\delta_{q,1}\delta_{m_s,-1/2}\delta_{m_s',1/2}+\delta_{q,-1}\delta_{m_s,1/2}\delta_{m_s',-1/2}\right)\end{bmatrix}, \tag{6b}$$

$$I_{qm_s'm_s} = \langle Im_I'|\hat{I}_1^q|Im_I\rangle = \hbar\begin{bmatrix} m_I\delta_{q',0}\delta_{m_I,m_I'} \\ +\frac{1}{\sqrt{2}}\left(-\delta_{q,1}\delta_{m_I,-1/2}\delta_{m_I',1/2}+\delta_{q,-1}\delta_{m_I,1/2}\delta_{m_I',-1/2}\right)\end{bmatrix}, \tag{6c}$$

and $\delta_{n,n'}$ is a Kronecker delta. The sum can be carried out using a computer program to arrive at

$$A_{n\ell jf} = \hbar^2 \left[\frac{\ell(\ell+1) - 3/4}{j\,(j+1)}\right] \left[\frac{f(f+1) - j(j+1) - 3/4}{2}\right] \left\langle r^{-3}\right\rangle . \tag{7}$$

Chap. 6. Normalization of potential well eigenfunctions.

It is actually pretty easy to normalize the bound state eigenfunctions of the finite potential well. First I do it for the even eigenfunctions, which can be written as

$$\psi_E^+(x) = N_E^+ \begin{cases} B^+ e^{\kappa_E^+ x} & x < -a/2 \\ \cos\left(k_E'^+ x\right) & -a/2 < x < a/2 \\ B^+ e^{-\kappa_E^+ x} & x > a/2 \end{cases} \tag{8}$$

where

$$k_E'^{\pm} = \frac{\sqrt{2mE'^{\pm}}}{\hbar} > 0, \tag{9}$$

$$\kappa_E^{\pm} = \frac{\sqrt{-2mE^{\pm}}}{\hbar} > 0, \tag{10}$$

and

$$E' = E + V_0. \tag{11}$$

Remember that the boundary conditions lead to

$$\cos\left(\frac{z'^+}{2}\right) = B^+ \exp\left(-\frac{z^+}{2}\right) \tag{12a}$$

$$k_E'^+ \sin\left(\frac{z'^+}{2}\right) = B^+ \kappa_E^+ \exp\left(-\frac{z^+}{2}\right) \tag{12b}$$

where

$$z^{\pm} = \sqrt{-\frac{2mE'^{\pm}}{\hbar^2}}a \tag{13}$$

$$z'^{\pm} = k_E'^{\pm} a = \sqrt{\beta^2 - \left(z^{\pm}\right)^2} \tag{14}$$

$$\beta^2 = \frac{2mV_0}{\hbar^2}a^2, \tag{15}$$

and that the energy is determined from

$$\tan\left(\frac{z'^+}{2}\right) = \frac{z^+}{z'^+}. \tag{16}$$

Using the wave function, it follows that the normalization is obtained from

$$2\left(N_E^+\right)^2\left[\int_0^{a/2} dx\cos^2\left(z'^+x/a\right)+\left(B^+\right)^2\int_{a/2}^{\infty} dx e^{-2z^+x/a}\right]=1$$

$$2\left(N_E^+\right)^2\left[\int_0^{a/2} dx\cos^2\left(z'^+x/a\right)+\cos^2\left(z'^+/2\right)e^{z^+}\int_{a/2}^{\infty} dx e^{-2z^+x/a}\right]=1$$

$$\left(N_E^+\right)^2\left[\frac{z'^+ + \sin z'^+}{z'^+}+\frac{2\cos^2\left(z'^+/2\right)}{z^+}\right]=\frac{2}{a}$$

$$\left(N_E^+\right)^2\left[1+\frac{2\sin\left(z'^+/2\right)\cos\left(z'^+/2\right)}{z'^+}+\frac{2\cos^2\left(z'^+/2\right)}{z^+}\right]=\frac{2}{a}.$$

From Eq. (16) and the fact that

$$\left(z'^{\pm}\right)^2+\left(z^{\pm}\right)^2=\beta^2$$

it follows that

$$\sin\left(\frac{z'^+}{2}\right)=\frac{z^+}{\beta};\qquad \cos\left(\frac{z'^+}{2}\right)=\frac{z'^+}{\beta}\ , \tag{17}$$

such that

$$\left(N_E^+\right)^2\left[1+2\frac{z^+}{\beta^2}+\frac{2z'^{+2}}{z^+\beta^2}\right]=\frac{2}{a};$$

$$N_E^+=\sqrt{\frac{2}{a}}\sqrt{\frac{z^+}{2+z^+}}.$$

The normalization now depends on energy.

Similarly for the odd parity states

$$\psi_E^-(x)=N_E^-\begin{cases} B^-e^{\kappa_E^- x} & x<-a/2 \\ A^-\sin\left(k_E'^- x\right) & -a/2<x<a/2 \\ -B^-e^{-\kappa_E^- x} & x>a/2 \end{cases}\ , \tag{18}$$

$$\sin\left(\frac{z'^-}{2}\right) = B^-\exp\left(-\frac{z^-}{2}\right) \tag{19a}$$

$$z'^-\cos\left(\frac{z'^-}{2}\right) = -B^-z^-\exp\left(-\frac{z^-}{2}\right) \tag{19b}$$

$$\tan\left(\frac{z'^-}{2}\right)=-\frac{z'^-}{z^-} \tag{20}$$

$$2\left(N_E^-\right)^2\left[\int_0^{a/2} dx \sin^2\left(z'^- x/a\right) + \left(B^-\right)^2 \int_{a/2}^{\infty} dx e^{-2z^- x/a}\right] = 1$$

$$2\left(N_E^-\right)^2\left[\int_0^{a/2} dx \sin^2\left(z'^- x/a\right) + \sin^2\left(z'^-/2\right) e^{z^-} \int_{a/2}^{\infty} dx e^{-2z^- x/a}\right] = 1$$

$$\left(N_E^-\right)^2\left[\frac{z'^- - \sin z'^-}{z'^-} + \frac{2\sin^2\left(z'^-/2\right)}{z^-}\right] = \frac{2}{a}$$

$$\left(N_E^-\right)^2\left[1 - \frac{2\sin\left(z'^-/2\right)\cos\left(z'^-/2\right)}{z'^-} + \frac{2\sin^2\left(z'^-/2\right)}{z^-}\right] = \frac{2}{a}$$

$$\left(N_E^-\right)^2\left[1 + \frac{2z^-}{\beta^2} + \frac{2z'^{-2}}{z^-}\right] = \frac{2}{a}$$

or

$$N_E^- = \sqrt{\frac{2}{a}}\sqrt{\frac{z^-}{2+z^-}}.$$

Chap. 6. Δx and Δp for a shallow well

I claimed that $\Delta x \gg a$ for a shallow well. In that case there is a single, even parity eigenfunction with $z^+ = \beta^2/2 \ll 1$ and $z'^+ \approx \beta$. Using the normalized wave function, I calculate

$$\begin{aligned}\left\langle x^2\right\rangle &= 2\frac{2}{a}\frac{z^+}{2+z^+} \\ &\quad \times \left[\int_0^{a/2} dx\, x^2 \cos^2\left(z'^+ x/a\right) + \frac{z'^{+2}}{\beta^2} e^{z^+} \int_{a/2}^{\infty} dx\, x^2 e^{-2z^+ x/a}\right] \\ &\approx 2\frac{z^+}{a}\frac{a^3}{4z^{+3}} = \frac{a^2}{2z^{+2}} = \frac{2a^2}{\beta^4}\end{aligned}$$

and

$$\begin{aligned}\left\langle p^2\right\rangle &= -\hbar^2\frac{4}{a^3}\frac{z^+ z'^{+2}}{2+z^+} \\ &\quad \times \left[-\int_0^{a/2} dx\, \cos^2\left(z'^+ x/a\right) + \frac{z^{+2}}{\beta^2} e^{z^+} \int_{a/2}^{\infty} dx\, e^{-2z^+ x/a}\right] \\ &\approx \frac{2\hbar^2 z^+ \beta^2}{a^3}\left(\frac{a}{2} - \frac{z^+ a}{2\beta^2}\right) \approx \frac{\hbar^2\beta^2}{a}\left(\frac{\beta^2}{2a} - \frac{\beta^2}{4a}\right) = \frac{\hbar^2\beta^4}{4a^2},\end{aligned}$$

so

$$\Delta x^2 \Delta p^2 \approx \frac{\hbar^2}{2} > \frac{\hbar^2}{4}$$

and

$$\frac{\Delta p^2}{2m} \approx \frac{1}{4}\beta^2 V_0 \ll V_0.$$

The fluctuations in kinetic energy are not sufficient to free the particle from the well. Actually the result corresponds to

$$\frac{\Delta p^2}{2m} \approx \frac{1}{4}\beta^2 V_0 = -E,$$

so the fluctuations in the kinetic energy are of order $|E|$.

Note that I can also calculate the average value of the energy, which should be E:

$$\langle E\rangle = \frac{\langle p^2\rangle}{2m} + \langle V\rangle$$

and

$$\begin{aligned}\langle V\rangle &= -2\frac{2}{a}\frac{z^+}{2+z^+}\int_0^{a/2} dx\, V_0 \cos^2\left(z'^+ x/a\right) \\ &\approx -V_0 z^+ = -V_0\beta^2/2 = 2E\end{aligned}$$

so

$$\frac{\langle p^2\rangle}{2m} + \langle V\rangle = -E + 2E = E.$$