Phys 511 Prof. P. Berman
Quantum Mechanics I Fall, 2003

Problem Set 4
Due Friday, October 3
No class Monday, October 6
The midterm exam is scheduled for Monday, October 20 from 5:30-7:00. Room 340 West Hall.

1. Consider a particle of mass m in a one-dimensional, infinite square well potential having length L. Find the energy levels and normalized eigenstates. At $t = 0$, the particle is in a linear superposition of the two lowest energy states

$$\psi(x,0) = 0.6\psi_1(x) + 0.8\psi_2(x)$$

Find the wave function for all time. Take $L = 1$ in some arbitrary units and plot $|\psi(x,t)|^2$ as a function of $\tau = E_1 t/\hbar$ for $\tau = 0, 0.1, 0.5, \pi, 2\pi$. What is the probability to be in state 1 as a function of time?

Which quantum state would correspond most closely to a classical particle having mass $m = 1$ gm and speed $v = 1$ cm/s if $L = 1$ cm?

2. Using box normalization, find $\psi(\mathbf{r},t)$ for a free particle if $\psi(\mathbf{r},0) = \left(\frac{1}{\pi\sigma^2}\right)^{3/4} e^{-r^2/2\sigma^2}$. Show that as the box size goes to infinity, you recover the result we have obtained previously and that the size of the quantization volume drops out of the calculation.

3. For the finite potential well considered in class, obtain a graphical solution for the odd parity eigenenergies. For what values of β^2 will there be exactly two bound odd parity states.

4. Consider an arbitrary 1-D potential that is always negative and vanishes for $x > |a|$. Assume the potential $V(x)$ is smooth, has no relative maxima, and $\int_{-\infty}^{\infty} V(x)dx$ is finite. Moreover the potential is weak in the sense that $\beta^2 = (2m/\hbar^2)V_0 a^2 \ll 1$, where $-V_0$ is a typical potential depth. Prove that this potential supports a bound state as β^2 goes to zero and find the energy of this bound state. (Hint: assume the bound state energy $-E \ll V_0$. Neglect the particle's energy in the classically allowed region and integrate the Schrödinger equation between $\pm b$, where b is a point in the classically forbidden region such that $b \gg a$ but $\sqrt{-(2m/\hbar^2)E}\, b \ll 1$.) You should find that the energy E is equal to $-\beta^2 V_0$, where V_0 can be defined through $-\int_{-a}^{a} V(x)dx = 2V_0 a$.)

5-6. Consider scattering by the potential $V(x) = -V_0$ for $-a \le x \le a$ and is zero otherwise. The transmission coefficient for this problem is given in Merzbacher, Eq. 6.63. You need not rederive this result in detail, but give the equations for matching boundary conditions (you can use Mathematica's **Solve** command to solve the equations if you wish**).** Show that the equation in Merzbacher can be written as

$$T = \frac{1}{1 + \left(\frac{n^2-1}{2n}\right)^2 \sin^2(2k'a)} = \frac{1}{1 + \frac{1}{2}\left(\frac{n^2-1}{2n}\right)^2 - \frac{1}{2}\left(\frac{n^2-1}{2n}\right)^2 \cos(4k'a)},$$

where $n = k'/k = \sqrt{(E+V_0)/E}$. This equation has the same form as that in optics for transmission from vacuum into a medium having index n and back into vacuum, where the medium has thickness $2a$. Show that maxima and minima (approximately) in transmission can be given a simple physical interpretation. How does the quantum result differ from the optical one?

b) Now consider a wave packet incident on the barrier. Suppose we do not wish to energy resolve the peaks and send in a wavepacket whose initial extent is much less than $2a$, but is sufficiently large to neglect spreading. In this case, there can be no interference in the scattering from the two boundaries. Why? Using the fact that the transmission coefficient at a boundary is $\frac{4n}{(n+1)^2}$ and the reflection coefficient is $\frac{(n-1)^2}{(n+1)^2}$, prove that the overall transmission coefficient is $T = \frac{2n}{(n^2+1)}$. Show that this agrees with the value of the transmission coefficient given above, averaged over the phase factor $\theta = 4k'a$; that is calculate $\bar{T} = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{1+\frac{1}{2}\left(\frac{n^2-1}{2n}\right)^2-\frac{1}{2}\left(\frac{n^2-1}{2n}\right)^2\cos(\theta)}$ which will average out all interference effects.